DOI: 10.17587/mau.22.3-11

V. P. Berdnikov, Postgraduate Student, berdnikov_vp@mail.ru,
V. M. Lokhin, D. Sc, Professor, kpu-mirea@yandex.ru,
MIREA — Russian Technological University, Moscow, 119454, Russian Federation

Corresponding author: **Berdnikov Vasiliy P.,** Postgraduate Student,
MIREA — Russian Technological University, Moscow, 119454, Russian Federation,
e-mail: berdnikov_vp@mail.ru

Accepted on October 5, 2020

Control of Parametrically Perturbed Objects with a Full Information

Abstract

The objective of this paper was to justify the new synthesis method of stabilizing controller for parametrically perturbed systems, which often appear in mobile robots, aircrafts, engineering objects with non-stationary parameters, intellectual control systems with a self-learning etc. Due to the high complexity and uncertainty of these systems, the classical PID controllers are not applicable and so a full information about the object state vector is used. Controllers obtained in this way allow to minimize the integral quality criterion of the system with the worst case parameter perturbation. For this purpose, the methods of differential games and switching systems theories were applied. Control laws are calculated by using the value function of the corresponding differential game, which can be obtained by solving the Hamilton-Jacobi-Bellman-Isaacs equations. A special set of basic functions was developed to approximate the value function and satisfy the boundary conditions. Finally, controller synthesis for a specific object with a nonstationary parameter is given. It significantly exceeds both the linear and fuzzy controllers in terms of quality. In the task of analyzing system qualitative characteristics under the worst parametric perturbation, our results are compared to the modern direct collocation methods of optimal control. With the same accuracy, proposed method is two times faster for low order systems. To verify that developed controllers can be employed in real time applications, we present computational time and memory usage in the end of the article.

Keywords*: differential games, Hamilton-Jacobi-Bellman-Isaacs equations, value function, switching systems, absolute stability, Lyapunov functions, implicit Euler scheme, fuzzy controllers*

Acknowledgments. This work was supported by the Russian Science Foundation, according to research project No 16-19-00052. Also, authors would like to thank Natalia Sadivova for translation assistance throughout all aspects of this paper.

For citation:

Berdnikov V. P., Lokhin V. M. Control of Parametrically Perturbed Objects with a Full Information, *Mekhatronika, Avtomatizatsiya, Upravlenie,* 2021, vol. 22, no. 1, pp. 3—11.

DOI: 10.17587/mau.22.3-11

УДК 681.511.4

DOI: 10.17587/mau.22.3-11

В. П. Бердников, аспирант, berdnikov_vp@mail.ru,
В. М. Лохин, д-р техн. наук, проф., kpu-mirea@yandex.ru,
ФГБОУ ВО МИРЭА — Российский технологический университет, Москва, Россия

Управление параметрически возмущаемыми объектами при наличии полной информации

Целью данной статьи является обоснование нового метода синтеза стабилизирующих регуляторов для параметрически возмущенных систем, которые часто встречаются в мобильной робототехнике, беспилотных летательных аппаратах, исполнительных приводах с нестационарными параметрами, интеллектуальных системах управления с самообучением и т. д. Из-за высокой сложности и неопределенности этих систем классические ПИД регуляторы *оказываются неприменимы, поэтому в данной работе предлагается использовать полную информацию о векторе состояния объекта. Полученные таким образом регуляторы позволяют минимизировать интегральный критерий качества системы при наихудшем возмущении ее параметров. Для этого были применены методы дифференциальных игр и теории переключаемых систем. Законы управления вычисляются на основе функции цены соответствующей дифференциальной игры, которая может быть получена путем решения уравнений Гамильтона—Якоби—Беллмана—Айзекса. Для аппроксимации функции цены и удовлетворения граничных условий был разработан специальный набор базисных функций. В последнем разделе приведен пример синтеза регулятора для конкретного объекта с нестационарным параметром. По качеству переходных процессов он значительно превосходит линейные и нечеткие регуляторы. В задаче анализа качественных характеристик системы при действии наихудших параметрических возмущений наши результаты сравниваются с современными численными методами оптимального управления. При той же точности предлагаемый метод работает в два раза быстрее для систем невысокого порядка. Чтобы убедиться, что разработанные регуляторы можно использовать в реальных системах, в конце статьи приводится время вычислений управляющих воздействий и объем использованной памяти ЭВМ.*

Ключевые слова: *дифференциальные игры, уравнения Гамильтона—Якоби—Беллмана—Айзекса, функция цены, переключаемые системы управления, абсолютная устойчивость, функции Ляпунова, неявная схема Эйлера, нечеткие регуляторы*

Introduction

In recent decades artificial intelligence methods have proven their efficiency in the tasks of complex dynamical objects control [1, 2]. One example of their successful application are non-stationary systems exposed to various uncertainty factors [3, 4]. These types of systems are used to describe aircrafts [5], mobile robots [6—8], actuators [9], etc. However, the choice of the structure and parameters of an intelligent controller as well as the analysis of the stability of the overall system is still a difficult and not fully resolved problem. A possible solution is the theory of absolute stability, which allows to determine the admissible sectors in which the nonlinear characteristics of the intelligent controller should be located.

Numerical algorithm proposed in Berdnikov (2018) studies can be used for constructing Lyapunov spline functions to solve the problem of absolute stability of the system with several nonstationary elements [10]. Later, this algorithm was used to construct guaranteed stability regions of automatic control system (ACS) with fuzzy controllers and parametrically perturbed objects [11, 12]. Even though the synthesized systems are stable, two questions still need to be answered. The first one is related to the choice of a fuzzy controller specific characteristics in a stable sector. The second one concerns determination of the object's worst-case parameters changing.

To avoid these drawbacks, this paper proposes the new algorithm for synthesizing nonlinear controller characteristics of ACS with parametrically perturbed objects. Its main idea is to use the theory of differential games for switching systems. In this context, a two-player differential game is considered. One of the players tries to minimize a preselected quality criterion by using control, while the other tries to maximize it using parametric perturbation. Approximation of the game value function is the result of the algorithm. If there is complete information about the state vector of the system, this approach allows to synthesize optimal control laws in the form of feedbacks. On the other hand, the analysis of differential game permits the formation of the worst-possible perturbations in the system. It leads to the guaranteed estimation of the system quality criterion, which means it can't be worse with any other possible parameters perturbations.

This work is structured in the following way. Section 1 includes a detailed task description, the aim of which is to resolve a Hamilton-Jacobi-Bellman-Isaacs partial differential equation (HJBI). Section 2 introduces the special set of basis functions which help to approximate solution of the HJBI equations. Distinguishing characteristic of these functions is that the boundary condition is always satisfied. In Section 3, to simplify the process of finding a solution, methods of differential games with switching strategies are applied. Section 4 develops a numerical algorithm for constructing an approximate solution of HJBI equations for a switching system. Section 5 gives an analysis of the specific system with parametrically perturbed object. Discussion of the results and conclusion are presented in Section 6.

1. Problem statement

Consider a system described by equations

$$\frac{dx}{dt} = F(t, x, u, w) = f(x) + g_u(x)u + g_w(x)w, \quad (1)$$

where $x = (x_1, x_2, \ldots, x_d)^{\mathrm{T}}$ — d-dimensional column vector of state variables, $u = (u_1, u_2, \ldots, u_p)^{\mathrm{T}}$ —

p-dimensional column vector of controls, $(w, w_2, \ldots, w_q)^{\mathrm{T}}$ — q-dimensional column vector of disturbance; $f(x)$, $g_u(x)$, $g_w(x)$ — matrix functions of the corresponding dimensions. The restrictions on the vectors of control and disturbances $u \in U$, $w \in W$ (2) are imposed.

$$U = \{u \in \mathbb{R}^p | -\infty < u_i^1 \leqslant u_i \leqslant u_i^2 < +\infty\};$$
$$W = \{w \in \mathbb{R}^q | -\infty < w_j^1 \leqslant w_j \leqslant w_j^2 < +\infty\}. \quad (2)$$

This work examines the issue of synthesis of stabilizing optimal controller for parametrically perturbed objects. In contrast to [11, 12], this type of controller should minimize some pre-selected quality criterion J. Obviously, it will be different for various perturbations of a given class $w(t)$. In this regard, it is advisable to tune the controller, assuming that the system has some "worst" $w^*(t)$ perturbation for the chosen criterion, i.e. the perturbation is maximizing J among all other $w(t)$ (in the context of the concept of antisipative strategies [13]).

It is well known from the theory of differential games [14] that if the quality criterion is presented as the sum of the integral component and the terminal cost:

$$J(x_0, u(\cdot), w(\cdot)) = \int_0^T L(x, u, w)dt + h(x(T)),$$
$$x_0 = x(0),$$

the solution of the partial differential equation (3) $V(t, x)$,

$$\frac{\partial V}{\partial t} + \min_{u \in U} \max_{w \in W} \left\{ \frac{\partial V}{\partial x}, F(t, x, u, w) + L(t, x, u, w) \right\} = 0, V(T, x) = h(x), \quad (3)$$

when the additional condition of the saddle point is fulfilled (Isaac's condition)

$$\min_{u \in U} \max_{w \in W} \{J(x, u(\cdot), w(\cdot))\} = \max_{w \in W} \min_{u \in U} \{J(x, u(\cdot), w(\cdot))\},$$

is the value function of the game. Moreover, if $V(t, x)$ known, the optimal control and the worst perturbation can be found in the form of feedback (4).

$$\begin{pmatrix} u(t,x) \\ w(t,x) \end{pmatrix} \in \arg\min_{u \in U} \max_{w \in W} \left\{ \left(\frac{\partial V}{\partial x}, F(x, u, w) \right) + L(x, u, w) \right\}. \quad (4)$$

Next, we will consider quality criteria which operates on an infinite time interval and do not explicitly depends on control and disturbance, namely:

$$J(x_0, u(\cdot), w(\cdot)) = \int_0^\infty L(x)dt, \quad (5)$$

where $L(x) > 0$ for $x \neq 0$ and $L(0) = 0$. It was shown in [15, 16] that if $V(x)$ is a solution of the equation

$$\min_{u \in U} \max_{w \in W} \left\{ \left(\frac{\partial V}{\partial x}, F(x, u, w) \right) + L(x, u, w) \right\} = 0,$$
$$V(0) = 0, \quad (6)$$

and if $V(x) > 0$, $dV/dt = (\partial V/\partial x, F(x, u, w)) < 0$ for $x \neq 0$, and the Isaac's condition is satisfied, then $V(x)$ is the Lyapunov function establishing the asymptotic stability of the system. Moreover, in some region of the origin $V(x)$ is the value function of the corresponding differential game, and the optimal controls and the worst perturbations are obtained in the form of feedback by the formula (4). Note that in our case, the Isaac's condition is satisfied, since system (1) is divided by control and perturbation, and the quality criterion does not at all depends on u and w [17].

Thus, calculation of $V(x)$ in (6) permits to solve the problem completely. This paper proposes a numerical method for approximation of the solutions of HJBI equations, which contains the following main steps:

1) Construction of a set of basis functions for which the boundary condition $V(0) = 0$ is always satisfied;

2) Transformation of the initial system into a switching one. This step will simplify the subsequent procedure of the numerical search for a solution (6);

3) Development of numerical algorithm for approximating the value function $V(x)$ of a switching system.

In the following sections, the steps set out above will be discussed.

2. Basis Functions Construction

There are several numerical methods of approximation for the solutions of nonlinear partial differential equations in current literature. They include finite-difference methods [18], finite-elements methods [19], Galerkin methods [20], methods based on radial basis functions (RBF) [21] and others [see, for example, 22]. We will consider the last one. The main advantage of using radial basis functions is that it is not necessary to define a dense structured space grid, at the nodes of which the desired solution is approximated. This circumstance makes it possible to better cope with the so-called curse of dimensionality," namely that the computational resources and computer memory exponentially grow with increasing dimension d. In fact, classical numerical methods are not applicable for the dimension $d \geqslant 4$ of the space on non-specialized computers.

In RBF-based methods, the solution is searched in the form

$$S(x) = \sum_{k=1}^{m} a_k \varphi_k(x), \quad (7)$$

where $\varphi_k(x)$ — are basis functions, m is the number of basis functions, a_k are the coefficients that need to be found. Radial basis functions are defined as follows:

$$\varphi_k(x) = g(p_k - x),\ p_k \in \mathbb{R}^d.$$

Function $g(\cdot)$ and the centers of the basis functions p_k are selected according to the specifics of a particular problem [23].

One of the main issues in the numerical solution of HJBI is that the functions $V(x)$ in (6) are almost always non-smooth [24, 25]. That's why it requires the use of an excessively large number of basis functions to approximate with acceptable accuracy. From a practical point of view, this circumstance makes methods based on radial basis functions inapplicable for the analysis of systems in which the control object is described by even a low order equations. In order to find solutions, it is necessary to take into account the boundary conditions (6), which complicate the structure solution and calculation process of a_k in (7).

To avoid such difficulties, this paper introduces a special set of basis functions, for which, when exercised, the boundary conditions $V(0) = 0$ in (6) will be satisfied automatically. The final non-smooth solution will be formed by several smooth surfaces. An overview of the proposed functions is given below:

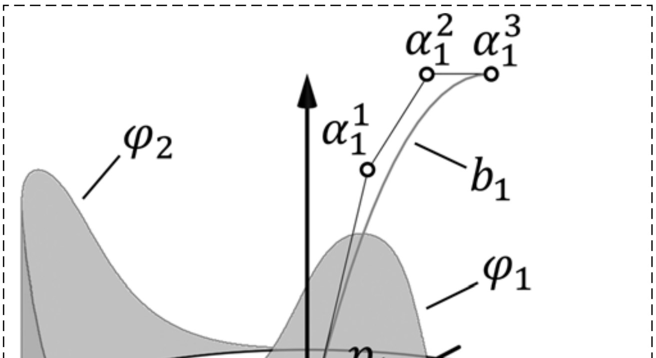


Fig. 1. Example of a set of basis functions on the plane

$$S(x) = \sum_{i=1}^{m} b_i^l \left(\frac{r}{r_{\max}}\right) \varphi_i \left(\frac{x}{r}\right),\ r = \|x\|,\ r_{\max} > 0\,, \quad (8)$$

$b_k^l(\cdot)$ is the Bezier spline of order l, and $r_{\max}$ defines the radius of the region in which the solution is sought. The paper furthermore uses

$$\varphi_i(y) = \left(\frac{\langle y, p_i\rangle + 1}{2}\right)^{\frac{1}{\mu^2}},\ \ \|p_i\| = 1, 0 < \mu < 1,$$

as a function $\varphi_i(\cdot)$, although there may be another option. For certainty we will assume that $\varphi_i(x/r) = 0$ for $r = 0$. Parameter μ allows to change the shape of $\varphi_i(\cdot)$: as μ increases, the region in which $\varphi_i(\cdot)$ is significantly different from zero and vice versa. Represented Bezier spline $b_k^l(\cdot)$ in the form of the Bernstein polynomial (8) can be rewritten as follows

$$\begin{aligned} S(x) &= \sum_{i=1}^{m} \sum_{j=0}^{l} a_i^j B_j^l \left(\frac{r}{r_{\max}}\right) \varphi_i \left(\frac{x}{r}\right); \\ B_j^l(z) &= \frac{l\,!}{j\,!(l-j)!} z^j (1-z)^{l-j}. \end{aligned} \quad (9)$$

Thus, the construction of smooth surfaces tends to calculate the coefficients a_i^j. Note that if we take $a_i^0 = 0$ (for all $i = 1, ..., m$), then $S(0) = 0$. Therefore, when using functions (9) as an approximation of the solution (6), the condition $V(0) = 0$ in (6) will be satisfied automatically, without introducing special restrictions. In the next section, functions of the form (9) will be used to construct smooth sections of the value function. Fig. 1 shows a graphical representation of the proposed functions, where, for convenience, $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are located on circles of different radius.

3. Transformation into a switching system

It follows directly from the equation (6) that the optimal control u and the worst perturbation w can only take boundary values, since the quality criterion (5) does not depend explicitly on u, w. Therefore, the entire domain Ω in which the solution of the equation (6) is sought can be divided into subdomains $\Omega^{a,b}$ with both constant control and perturbation. When the system trajectory moves from one subdomain to another, u and w switch instantly. As will be shown below with numerical examples, the solution in each $\Omega^{a,b}$ can be approximated with high accuracy using the surface (9).

If the boundaries $\Omega^{a,b}$ are known, the search for the value function can be reduced to the problem of solving a system of linear equations. However, it is generally scarcely possible to determine $\Omega^{a,b}$ without solving the HJBI. In fact, the task of synthesizing the optimal ACS is reduced to the following question: "In which areas of the phase space of the system do these or other values of control act?" In this regard, we use the results of the theory of differential games with switching strategies to construct the solution (6).

It was proved in [26, 27] that the solution $V(t, x)$ of partial differential equations (3) can be obtained as the limit

$$\lim_{n\to\infty} V_n^{a,b}(t,x) = V(t,x),$$

where $V_n(t,x) = [V_n^{a,b}(t,x)]$ is a continuous matrix function, which is the viscosity solution of the system

$$\max\left\{\begin{matrix}\min\left\{\begin{matrix}\dfrac{\partial V_n^{a,b}(t,x)}{\partial t} + H^{a,b}\left(t,x,\dfrac{\partial V_n^{a,b}(t,x)}{\partial x}\right), \\ M^{a,b}[V_n](t,x) - V_n^{a,b}(t,x)\end{matrix}\right\}, \\ M_{a,b}[V_n](t,x) - V_n^{a,b}(t,x)\end{matrix}\right\} = 0;$$

$$\begin{aligned}&V_n^{a,b}(T,x) = h(x);\\ &M^{a,b}[V_n](t,x) = \min_{\tilde{a}\neq a}\{V_n^{\tilde{a},b}(t,x) + \gamma_n\};\\ &M_{a,b}[V_n](t,x) = \max_{\tilde{b}\neq b}\{V_n^{a,\tilde{b}}(t,x) - \mu_n\};\\ &H^{a,b}(t,x,p) = p, F(t,x,a,b) + L(t,x,a,b);\\ &a \in U_n \subset U, b \in W_n \subset W.\end{aligned} \tag{10}$$

Here γ_n, μ_n are the costs for switching between different controls a and perturbations b, which tend to zero as $n \to \infty$. Finite sets U_n, W_n cover the initial sets of controls and perturbations, and as n increases, the coverage should become more and more dense. Since u, w take only boundary values in the task described, the discretization of the sets U, W occurs naturally. Therefore, U_n and W_n consist of 2^p and 2^q elements and, consequently, are independent of n. If a solution (10) is found, then $V_n(t, x)$ is the lower value function of the differential game with payoff functional

$$J(x_0,u(\cdot),w(\cdot)) = \int_0^T L(t,x(t),u(t),w(t))\,dt + \\ + h(x(T)) + N_{a(\cdot)}\gamma_n - N_{b(\cdot)}\tau_n,$$

in which $N_{a(\cdot)}$, $N_{b(\cdot)}$ is understood as the number of control and disturbance switches in the interval [0, T].

Despite the fact that to solve (10) it is necessary to calculate $2^{(p+q)}$ functions $V_n^{a,b}(t,x)$, this approach has certain advantages over direct computation of $V(t, x)$ in (3). Indeed, the dependence on the gradient of $V(t, x)$ in (3) is non-linear, moreover, in many practical cases it is not smooth, and (3) is neither convex nor concave in $\partial V(t,x)/\partial x$. At the same time, in (10) $\partial V_n^{a,b}(t,x)/\partial x$ enter linearly. As will be shown in the next section, it allows to compute $V_n(t, x)$ by the sequential solving systems of linear equations. Thus, there are 2 main assets in favor of using (10) instead of (3): discreteness of the control and disturbances sets also as simplifying the procedure of calculating the value function.

4. Formation of the optimization problem

On the time interval [0, T], we introduce $M + 1$ uniformly distributed discrete values $\tau_M < \dots \tau_{i+1} < < \tau_i < \tau_{i-1} < \dots < \tau_0$, with $\tau_0 = T$, and $\tau_M = 0$. We also discretize the state space $X \subset \mathbb{R}^d$ using N (not necessarily structured) points $x_j \in X$. Then, using implicit Euler scheme, we will search for the solution (10) in the following form:

$$V_{i+1,j}^{a,b} - \Delta t H^{a,b}\left(t,x,\frac{\partial V_{i+1,j}^{a,b}}{\partial x}\right) = \underline{V}_{i+\frac{1}{2},j}^{a,b}, \tag{11}$$

where

$$\begin{aligned}&\underline{V}_{i+\frac{1}{2},j}^{a,b} = \min\left[\bar{V}_{i+\frac{1}{2},j}^{a,b}, \min_{\tilde{a}\neq a}\left\{\bar{V}_{i+\frac{1}{2},j}^{\tilde{a},b} + \gamma_n\right\}\right];\\ &\bar{V}_{i+\frac{1}{2},j}^{a,b} = \max\left[V_{i,j}^{a,b}, \max_{\tilde{b}\neq b}\{V_{i,j}^{a,\tilde{b}} - \mu_n\}\right];\\ &V_{i,j}^{a,b} = V_n^{a,b}(t_i,x_j), V_{0,j}^{a,b} = h(x_j).\end{aligned} \tag{12}$$

In this work, the values of $V_{i+1,j}^{a,b}$ and $\partial V_{i+1,j}^{a,b}/\partial x$ are proposed to be approximated by using the function $S(x)$ from (9). By taking the total number of coefficients for the basis functions in (9) as K, and also setting $M > K$, we conclude that (11) is a linear equations system. Since it is overdetermined, its solution must be sought in the least squares sense.

Thus, the transition from t_i to t_{i+1} consists of two stages. At the first stage, the values $V_{i,j}^{a,b}$ used in (12) serve to calculate $\underline{V}_{i+1/2,j}^{a,b}$. The coefficients for the basis functions in (9) are determined at the second stage, while solving the linear equations system (11). When this procedure complete for all τ_i, we obtain a solution on the interval [0, T].

Now we will consider how this technique can be used to solve the stationary HJBI equation that arises in a differential game on an infinite time interval. To do this, it is necessary to conduct a series of numerical experiments, gradually increasing T. If, starting from some moment $T > T_c$, the value function stabilizes at the initial moment of time, i.e. $\partial V_n^{a,b}(0,x)/\partial t$ takes small values, then we can take

$$V(x) = \min_a \left[\max_b V_n^{a,b}(t_M, x) \right]. \qquad (13)$$

Note, that the function $V(x)$ and its gradient can be calculated not only at points x_j, but also at any other points $x \in X$ without additional interpolation. Indeed, the functions $V_n^{a,b}(t_M, x)$ are represented in the form (9) and the basis functions coefficients occur when (11) solved. If it is possible to obtain complete information about the state vector x for some control object at each moment of time, then the function (13) can be used directly on-board devices to synthesize optimal control laws.

In this way, the synthesis method of the optimal stabilizing controller consists of the following steps:

1. Equations of the system are written in the form (1);
2. Desired quality criterion (5) is selected;
3. Radius of the region $r_{\max}$ in which the solution is sought (6), the number of basis functions, and the parameter μ, which determines the shape of the basis functions, are specified;
4. Value function (13) is searched while solving the optimization problem described in this section;
5. If $V(x) = 0$ for $x \neq 0$, then the control law $u(x)$ from (5) is stabilizing and optimal with respect to the selected quality criterion.

The examples of using this method are presented below.

5. Numerical experiments

Consider an ACS with a parametrically perturbed object from [12], which structure is shown in Fig. 2.

Dynamic equations of the presented system have the following form:

$$\frac{dx}{dt} = Ax + b_p \varphi(\sigma_p, t) + b_w w(t)\sigma_w;$$

$$A = \begin{pmatrix} -\frac{1}{T_2} & \frac{1}{T_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{T_1} \end{pmatrix}, b_p = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{pmatrix},$$

$$b_w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \qquad (14)$$

$$\sigma_p = -x_1, \ \sigma_w = -x_2 + x_3;$$

$$\delta_p^1 \leqslant \frac{\varphi(\sigma_p, t)}{\sigma_p} \leqslant \delta_p^2, 1 = \delta_r^1 \leqslant w(t) \leqslant \delta_r^2 = 3.$$

If the controller is linear, then $\varphi(\sigma_p, t) = k_p \sigma_p$, $\delta_p^1 \leqslant k_p \leqslant \delta_p^2$. If the controller is fuzzy, then $\varphi(\sigma_p, t)$ determines the nonlinearity in the channel of the P-controller that satisfies the sector constraints δ_p^1, δ_p^2. The quality criterion is chosen in the form (5) with $L(x) = x_1^2 + x_2^2 + x_3^2$. It characterizes accumulated deviation of the system state and zero equilibrium point.

Next, we calculate the values of the selected quality criterion for three different types of P-controller: linear, fuzzy P-controller from [12] (its characteristics are demonstrated in Fig. 3), and controller with the control law $\xi(x) = k_p(x)\sigma_p$, $\delta_p^1 \leqslant k_p(x) \leqslant \delta_p^2$. The coefficient $k_p(x)$ of third controller is determined by complete information about the state vector.

The results of the calculation with the initial conditions $x_1(0) = 5$, $x_1(0) = x_3(0) = 0$ are summa-

Value of the quality criterion for different types of controllers

Controller Type	LC		FC		FIC		
Controller Parameters	$k_p = 5$	$k_p = 7$	$\delta_p^1 = 0.1$ $\delta_p^2 = 5.3$	$\delta_p^1 = 2.0$ $\delta_p^2 = 6.2$	$\delta_p^1 = 0.1$ $\delta_p^2 = 5.3$	$\delta_p^1 = 2.0$ $\delta_p^2 = 6.2$	$\delta_p^1 = 2.0$ $\delta_p^2 = 8.0$
Criterion Value	15.5	217.2	4.9	5.5	1.3	2.3	2.2

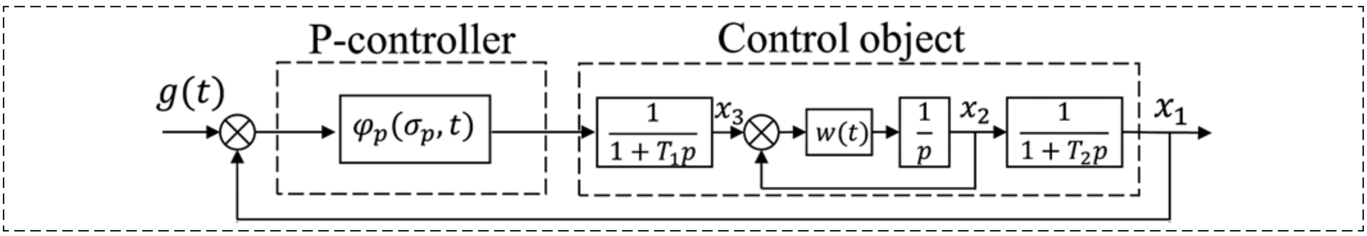
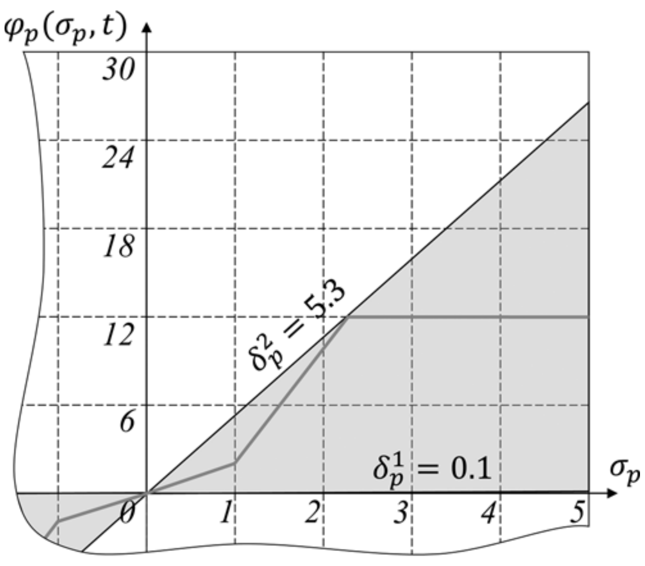


Fig. 2. ACS with a parametrically disturbed object



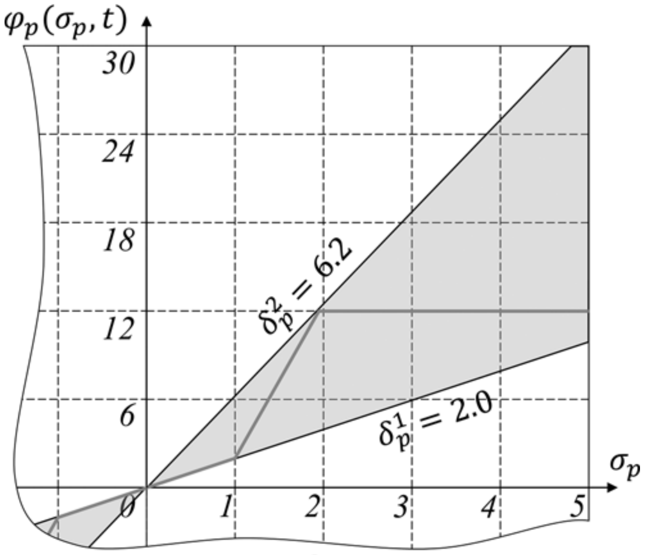


Fig. 3. Characteristics of the fuzzy P-controller for $\delta_p^1 = 0.1$, $\delta_p^2 = 5.3$ (*a*) and $\delta_p^1 = 2.0$, $\delta_p^2 = 6.2$ (*b*)

rized in Table (LC — linear controller, FC — fuzzy controller, FIC — controller with full information).

The rapid growth of the quality criterion J with an increase of k_p of the LC indicates the proximity to the stability boundary. Already at $k_p = 7.3$, the system is unstable (in this case, the value of the selected quality criterion tends to infinity). A FC with non-linear characteristics from [12] shows significantly lower values of J compared to the linear controller. However, really interesting results are associated with FIC. According to the same sector constraints as for the FC, the values of the criterion are 4 times lower for $\delta_p^1 = 0.1$, $\delta_p^2 = 5.3$ and 2.4 times lower for $\delta_p^1 = 2.0$, $\delta_p^2 = 6.2$. Moreover, if the possibilities of increasing the upper boundary of the sector δ_p^2 without changing $\delta_p^1, \delta_w^1, \delta_w^2$ are exhausted for the FC, then this is not the case for FIC, because in the last experiment the value of δ_p^2 was increased to 8.0, and the level of J was even lower than those of the controller with $\delta_p^2 = 6.2$. Note that for $k_p = 8.0$, the system with LC is unstable, i.e. at certain moments in time, the FIC switches to an unstable mode to improve quality index. The upper limit of change in the non-stationary parameter $w(t)$ can also be increased. Even with $\delta_w^2 = 9$, the value of J is only 3.1, which is lower than the linear and fuzzy P-controllers for $\delta_w^2 = 3$.

Determination of the worst parametric disturbances $w^*(t)$ in the system is another possible application of the proposed method. It allows to obtain information about the quality and stability of ACS in the most adverse events (a worst-case scenario). In most cases it is impossible to get analytic form of $w^*(t)$ and numerical algorithms must be applied. Fig. 4, *a* presents an example of ACS with a linear controller (k_p = 7.0) and an object, in which the parameter $w(t)$ changes in the worst way (based on our method).

As can be seen from the analysis of Fig. 4, *a*, strong oscillations are observed in the system. It indicates a system's proximity to the stability boundary and is completely coherent with [11], where this estimation was obtained independently on the basis of Lyapunov spline functions. In order to make sure that this disturbance is really the worst, another numerical method of optimal control will be applied [28]. It is a direct collocation method which reduces optimal control problem to nonlinear programming task. Since this work studies system on infinite time interval, sufficiently large process end time (t_f = 100) for numerical direct collocation method is chosen. It makes it possible to compare two processes on shorten start period $t \in [0, 10]$. The results of algorithm [28] are given in Fig. 4, *b*. They are similar to those in Fig. 4, *a*, and so, founded parametric disturbance is the worst. In contrast with our method, the resulting disturbance is given in time-dependent form (open loop form). This means that for different initial condition $x(0)$ direct collocation algorithm must be run again. Our method leads to feedback form of disturbance and only an approximation of the value function $V(x)$ is needed, what can be done once for a given system.

All experiments were done on two core 1.8 GHz CPU computer with 4 GB RAM. Average time of $V(x)$ approximation for the example in this section is 76.5 s and it is much faster than direct

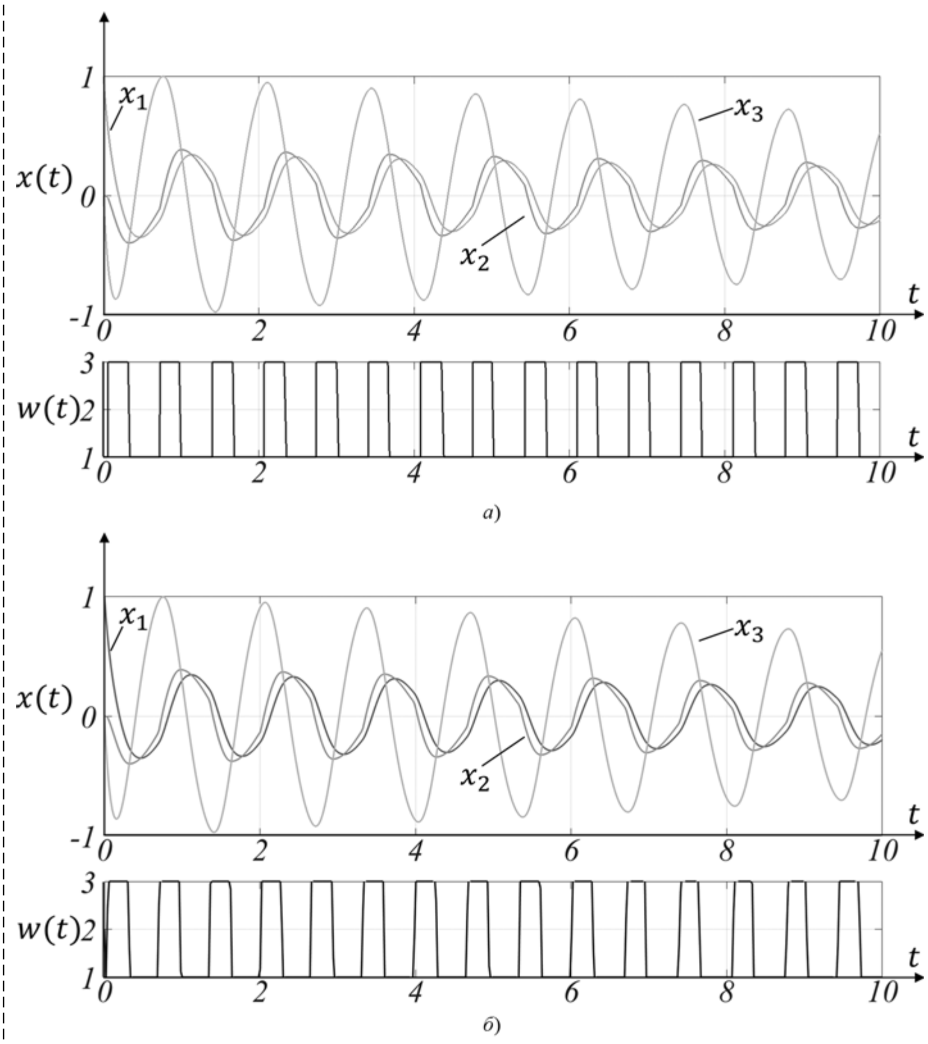


Fig. 4. Transient process in the system with a linear P-controller and the worst parameter change: proposed method (*a*), direct collocation method (*b*)

collocation algorithm which average time is 1704.6 s. Such difference can be explained in several ways. As mentioned above, the end time of process t_f must be taken large enough, and all this time $w(t)$ switches instantly between two boundary values. The same type of problem (bang-bang optimal control problem) is one of the hardest to solve in practical optimal control theory. To deal with a high number of switches, direct collocation algorithms need to subdivide time interval into many parts and precisely detect points of switching, what take a lot of computational resources. However, direct collocation algorithms do not suffer from "curse of dimensionality" and for high order systems it is still the only way to estimate the worst disturbances.

Since used computer is comparable with modern onboard systems for mobile robots and medium sized drones, it is interesting to discuss some aspects of physical implementations of proposed control algorithms. First, it requires a full information about object state vector x. In most cases this information can be obtained with a help of special sensors (for specified objects). Second, it requires enough onboard memory size to handle approximated value function. Third, it requires gradient computation of approximated value function at point x for every new time step. For 3-dimensional system (14) the approximation of $V(x)$ uses under 10 KB of memory and evaluation of gradient takes around 0.005 s. Nonetheless, it must be noted that these values depend on system dimension and will grow rapidly with an order of control object.

6. Conclusion

This paper proposes a method for synthesizing ACS for parametrically disturbed objects. It is shown that the stabilizing control law can be obtained in the form of feedback based on the value function of a particular differential game. The value function is calculated by solving the nonlinear Hamilton-Jacobi-Bellman-Isaacs equation. Since, in general, there is no analytical solution, a numerical procedure that approximates the value function was suggested. For this purpose, a novel set of basis functions was developed and methods of the theory of differential games with switching strategies were used to simplify the numerical procedure.

Suggested method allows to analyze asymptotic stability of parametric perturbed systems with high precision which is confirmed by Lyapunov functions. Also, it permits to find the worst parameters change in the system and thus helps to get guaranteed estimation of quality index. In comparison with direct collocation method it is more than 20 times faster. Meshfree nature of the proposed method leads to the efficient value function approximation in the sense of required memory size and gradient evaluation time. However, this method is not "curse of dimensionality" free, so achieved performance takes place only for low order systems.

Consequently, future works will be concentrated on studying higher order cases and improvements which need to be done to handle this type of problems. At the same time, physical implementation of control system for trajectory motion of mobile robots and quadcopters will be realized. One of the main features of this approach is the necessity of real-time full information about object state vector. On the one hand it will increase complexity of the system hardware, but on the other may give an advantage to the state of the art methods.

References

1. **Nowakova J., Pokorny M.** Intelligent Controller Design by the Artificial Intelligence Methods, *Sensors*, 2020, vol. 20 (16), pp. 1—27, DOI: 10.3390/s20164454.

2. **Santoso F., Garratt M. A., Anavatti S. G**. State-of-the-Art Intelligent Flight Control Systems in Unmanned Aerial Vehicles, *IEEE Transactions on Automation Science and Engineering*, 2018, vol. 15 (2), pp. 613—627, DOI: 10.1109/tase.2017.2651109.

3. **Vesely V., Korosi L.** Robust PI-D Controller Design for Uncertain Linear Polytopic Systems Using LMI Regions and H2 Performance, *IEEE Transactions on Industry Applications*, 2019, vol. 55 (5), pp. 5353—5359, DOI: 10.1109/tia.2019.2921282.

4. **Chesi G.** Convex Synthesis of Robust Controllers for Linear Systems With Polytopic Time-Varying Uncertainty, *IEEE Transactions on Automatic Control*, 2017, vol. 62 (1), pp. 337—349, DOI: 10.1109/tac.2016.2555701.

5. **Jafari M., Xu H.** Intelligent Control for Unmanned Aerial Systems with System Uncertainties and Disturbances Using Artificial Neural Network, *Drones*, 2018, vol. 2 (3), pp. 1—13, DOI: 10.3390/drones2030030.

6. **Wang G., Liu X., Zhao Y., Han S.** Neural Network-Based Adaptive Motion Control for a Mobile Robot with Unknown Longitudinal Slipping, *Chinese Journal of Mechanical Engineering*, 2019, vol. 32 (1), pp. 1—9, DOI: 10.1186/s10033-019-0373-3.

7. **Zheng W., Ito T.** Dynamic Surface Control—Based Adaptive Neural Tracking for Full-State Constrained Omnidirectional Mobile Robots, *Advances in Mechanical Engineering*, 2019, vol. 11 (4), pp. 1—14, DOI: 10.1177/1687814019846750.

8. **Guechi E.-H., Belharet K., Blazic S.** Tracking Control for Wheeled Mobile Robot Based on Delayed Sensor Measurements, *Sensors*, 2019, vol. 19 (23), pp. 1—21, DOI: 10.3390/s19235177.

9. **Bedioui N., Houimli R., Besbes M.** Simultaneous Sensor and Actuator Fault Estimation for Continuous-Time Polytopic LPV System, *International Journal of Systems Science*, 2019, vol. 50 (6), pp. 1290—1302, DOI: 10.1080/00207721.2019.1599078.

10. **Berdnikov V. P.** Improving the efficiency of the procedure of Lyapunov spline-functions construction for nonlinear nonstationary systems, *Russian Technological Journal*, 2018, vol. 6 (5), pp. 25—44 (In Russian), DOI: 10.32362/2500-316X-2018-6-5-25-44.

11. **Berdnikov V. P., Lokhin V. M., Uvaysov S. U.** Determination of Guaranteed Stability Regions of Systems with a PID Controller and a Parametrically Perturbed Control Object, *2019 International Seminar on Electron Devices Design and Production (SED)*, 23—24 April 2019. Prague, Czech Republic, DOI: 10.1109/SED.2019.8798412.

12. **Berdnikov V. P., Lokhin V. M.** Synthesis of Guaranteed Stability Regions of a Nonstationary Nonlinear System with a Fuzzy Controller, *Civil Engineering Journal*, 2019, vol. 5 (1), pp. 107—116, DOI: 10.28991/cej-2019-03091229.

13. **Plaskacz S.** Value functions in control systems and differential games: a viability method, Torun, Poland, Nicolaus Copernicus University, 2003, 126 p.

14. **Yong J.** Differential Games: A Concise Introduction, World Scientific, 2015, 322 p.

15. **L'Afflitto A.** Differential games, asymptotic stabilization, and robust optimal control of nonlinear systems, *IEEE Conf. Decision and Control*, 2016, vol. 2, pp. 1933—1938, DOI: 10.1109/CDC.2016.7798547.

16. **L'Afflitto A.** Differential games, continuous Lyapunov functions, and stabilization of non-linear dynamical systems, *IET Control Theory & Applications*, 2017, vol. 11 (15), pp. 2486—2496, DOI: 10.1049/iet-cta.2017.0271.

17. **Galperin E. A.** The cubic algorithm for global games with application to pursuit-evasion games, *Computers and Mathematics with Applications*, 1993, vol. 26 (6), pp. 13—31, DOI: 10.1016/0898-1221(93)90114-B.

18. **Wang S., Gao F., Teo K. L.** An upwind finite-difference method for the approximation of viscosity solutions to Hamilton-Jacobi-Bellman equations, *IMA Journal of Mathematical Control and Information*, 2000, vol. 17 (2), pp. 167—178, DOI:10.1093/imamci/17.2.167.

19. **Falcone M., Ferretti R.** Semi-Lagrangian schemes for Hamilton-Jacobi equations, discrete representation formulae and Godunov methods, *Journal of Computational Physics*, 2002, vol. 175, pp. 559—575, DOI: 10.1006/jcph.2001.6954.

20. **Beard R. W., McLain T. W., Wen J. T.** Successive Galerkin approximation of the Isaacs equation, *IFAC Proceedings Volumes*, 1999, vol. 32 (2), pp. 2071—2076, DOI:10.1016/s1474-6670(17)56351-x.

21. **Cecil T., Qian J., Osher S.** Numerical methods for high dimensional Hamilton—Jacobi equations using radial basis functions, *Journal of Computational Physics*, 2004, vol. 196 (1), pp. 327—347, DOI: 10.1016/j.jcp.2003.11.010.

22. **Falcone M., Makridakis C.** Numerical methods for viscosity solutions and applications, World Scientific Publishing Company, 2001, 249 p., DOI: 10.1142/4781.

23. **Fornberg B., Flyer N.** Solving PDEs with radial basis functions, *Acta Numer*, 2015, vol. 24, pp. 215—258, DOI: 10.1017/s0962492914000130.

24. **Subbotin A. I.** Generalized Solutions of First Order PDEs, Birkhäuser, Boston, 1995, 314 p., DOI:10.1007/978-1-4612-0847-1.

25. **Bardi. M., Capuzzo-Dolcetta I.** Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, Basel, 1997, 574 p., DOI: 10.1007/978-0-8176-4755-1.

26. **Yong J.** A zero sum differential game in a finite duration with switching strategies, *SIAM Journal on Control and Optimization,* 1990, vol. 28 (5), pp. 1234—1250, DOI: 10.1137/0328066.

27. **Yong J.** Differential game with switching strategies, *Journal of Mathematical Analysis and Applicationcs*, 1990, vol. 145 (2), pp. 455—469, DOI: 10.1016/0022-247X(90)90413-A.

28. **Nie Y., Faqir O. J., Kerrigan E. C.** ICLOCS2: Try this Optimal Control Problem Solver Before You Try the Rest, *2018 UKACC 12th International Conference on Control*, 5—7 September 2018, Sheffield, DOI: 10.1109/CONTROL.2018.8516795.