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Exponential Stability Regions Estimation of Nonlinear Dynamical Systems

Abstract

The main purpose of the research is the extending of the concept of qualitative exponential stability and instability for a wider class of dynamical systems and plants with estimating of regions of exponential stability as well as developing of analytic and calculating technologies for analyzing the quality of processes and projecting of control devices for control systems. And if the property of asymptotic stability indicates the convergence or divergence of the processes in time, the exponential stability provides information about the speed of convergence or divergence processes, thereby characterizing the rapidness of the system. Meeting the conditions of quality exponential stability evaluates the average rate of convergence or divergence of processes, as well as ongoing processes of deviations of the time-average behavior, the last gives information about the behavior of transients (oscillation, overshoot). Development of analytical and computational techniques for the analysis of stability and instability of comparison systems and, as a result of multiply-connected systems, as well as the processes quality is almost an essential task for the study multi-agent control algorithms and biologically inspired control algorithms in nonlinear systems, because the same systems should be related to each other and has a predetermined degree of exponential stability for the required control. Proposed results can be using in the developing flying and terrestrial robots based on biological control algorithm of living organism such as insects or bees.

Keywords: qualitative stability, Lyapunov function, sufficient conditions, performance, quality estimations, motion control, robotic complexes

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Оценка областей экспоненциальной устойчивости нелинейных динамических систем

Основной целью исследования является расширение понятия качественной экспоненциальной устойчивости и неустойчивости для широкого класса динамических систем и объектов управления с оценкой областей экспоненциальной устойчивости, а также разработка аналитических и расчетных способов анализа качества процессов и проектирование устройств управления для систем управления. И если свойство асимптотической устойчивости указывает на сходимость или расходимость процессов во времени, то экспоненциальная устойчивость дает информацию о скорости сходимости или расходимости процессов, тем самым характеризуя быстродействие системы. Удовлетворяя условиям качества экспоненциальной устойчивости, оцениваются средняя скорость сходимости или расходимости процессов, а также текущие процессы отклонений среднего по времени поведения, последний дает информацию о поведении переходных процессов (колебания, перерегулирование). Разработка аналитических и вычислительных методов анализа устойчивости и неустойчивости систем сравнения и, как следствие, многосвязных систем, а также качества процессов является практически необходимой задачей для исследования мультиагентных управляющих алгоритмов и алгоритмов управления в нелинейных системах, поскольку одни и те же системы должны быть связаны между собой и иметь заданную степень экспоненциальной устойчивости для требуемого управления. Предложенные результаты могут быть использованы при разработке летающих и наземных роботов на основе алгоритма биологического управления живым организмом, таким как насекомые или пчелы.

Ключевые слова: качественная устойчивость, функция Ляпунова, достаточные условия, производительность, оценки качества, управление движением, робототехнические комплексы

Introduction

Modern hardware of computers and computer technology analysis of the behaviour of multiply dynamic systems provides efficient algorithms for comparison of systems based on the Lyapunov functions, which are the result of the synthesis of multi-connected systems. The modular Lyapunov functions greatly simplify the procedures for the investigation of such system's behaviour. The terms of exponential and qualitative exponential stability and instability, obtained on the basis of modular Lyapunov functions, can judge the behaviour of processes and their quality of multiply-connected systems. Algorithms and software, developed on the basis of the results, obtained in [1, 9], allow to solve the problems of analysis and synthesis of continuous and discrete multi-connected systems with linear objects. The ideology worked out permit to consider qualitative exponential stability [9–13] and instability for systems and objects with continuous and discrete time similarly, and determine the sufficient local conditions to ensure these types of stability.

Problem Statement

Consider a discrete system, the movement of which is specified by the difference equation

$$x(m+1) = F(q(m, x(m)))x(m), \quad (1)$$

where m is the integer number of discrete intervals, x is n -dimensional state vector, $F(q(m, x(m)))$ is a square matrix of size $n \times n$, the elements of which depend on the changing values of the $q(m, x(m))$

vector, the component values which in turn depend on the number of discrete intervals and values state vector, and $q(m, x(m))$ is one-dimensional vector-valued function, continuous in each variable.

It is assumed that for arbitrary values of m and arbitrary values of the state vector $x(m) \in R^n$ values varying parameters are limited in the parameter space R^l by some simply connected closed domain D_q , i. e. $q(m, x(m)) \in D_q$ при $\forall m, \forall x(m) \in R^n$.

We define the quality estimation processes in the system (1) with the inequalities

$$\|x(m)\| \leq \rho \lambda^m \|x(0)\|; \quad (2)$$

$$\|x(m) - \beta^m x(0)\| \leq \rho (\lambda^m - \beta^m) \|x(0)\|, \quad (3)$$

where $\rho \geq 1$, the degree of attenuation $0 < \lambda < 1$, $\beta = \lambda - r$, where $0 < r \leq \lambda$. Note that if the inequality (2) is valid for (1) then the system is exponentially stable, and this inequality allows to measure system performance — evaluation of the transient time over the set of trajectories with initial values of the state vector $x(0)$ for which the value of the norm is constant.

The system is exponentially unstable if parameter $\lambda > 1$ in the inequality (2), while if more, and $1 < \beta + r < 1$ in the inequality (3), then the system is qualitatively exponentially unstable. The inequality (3) for the value $r = \lambda (\beta = 0)$ gives (2). The inequality (3) for (1) gives a local estimate of the behaviour of processes in the system, namely the estimate of the norm deviation from exponential decaying process $\beta^m (x(0) (\|\beta\| < 1))$, which provides a stabilization system's evaluation of the first release and overshoot over the set of trajectories ($\|x(0) = d\|$, where $d > 0$).

For (1) the following task is set: to find within parameter space R_l such a range of permissible changes of the parameters D_q , that for an arbitrary time changing settings, limited by the values of parameters within this range, quality estimations of (2) and (3) types are guaranteed for the system (1).

Sufficient Conditions for Meeting the Required Quality Estimations

Let's state sufficient conditions for meeting the required quality estimations (2) and (3). In order to satisfy the evaluations of quality processes (2) and (3) with values β and λ for the system (1) in the limited domain D_q with the values of the parameters $q(m, x(m))$, it is sufficient that such a positive definite symmetric matrix P size $n \times n$ exists, that for all values of $q \in D_q$ is valid the inequality

$$\bar{F}^T(q)P\bar{F}(q) - r^2P \leq 0 \quad (4)$$

which is understood in the sense of a negative semi-certainty of the resulting matrix in the left-hand part of inequality, where

$$\bar{F}(q) = F(q) - \beta I.$$

And the value $p = \sqrt{\frac{C_2}{C_1}}$, where C_2 and C_1 are maximum and minimum proper numbers of the matrix P .

Checking of the condition (4) is associated with the use of beam properties of quadratic forms. Consider the characteristic equation

$$\det[\bar{F}^T(q)P\bar{F}(q) - \mu P] = 0 \quad (5)$$

and assume that at a fixed value of q the maximum root $\mu + (q)$ is found.

Then the inequality

$$\mu^{1/2} + q \leq r \quad (6)$$

gives the validity of (4).

Note that the maximum root of the characteristic equation (5) coincides with the maximum eigenvalue of the matrix $\bar{F}^T(q)P\bar{F}(q)P^{-1}$. Thus, the calculation of the maximal root $\mu + (q)$, depending on the values of the parameter vector q with the following checking of the inequality (6), can form a base for establishing the boundaries of acceptable change settings. However, this method of determining the boundaries of $D_q(\beta, r)$ is effective only

when q is a scalar. If q is a vector quantity, then to simplify the calculations, we find further ellipsoidal estimation of the parameters acceptable change range.

We formulate such a condition for a continuous system, the movement of which is set by

$$\dot{x}(t) = F(q(t, x(t)))x(t) \quad (7)$$

where all variables and matrixes have the same meaning as in equation (1), moreover it is assumed that the matrix $F(q(t), x(t))$ is such that the solution of (7) for any initial conditions exists and is unique.

In a continuous system (7) the quality estimation like (2), (3) are defined like

$$\|x(t)\| \leq \rho e^{-\alpha t} \|x(0); \quad (8)$$

$$\|x(t) - e^{-\beta t} x(0)\| \leq \rho(e^{-\alpha t} - e^{-\beta t}) \|x(0) \quad (9)$$

if $\alpha > 0$, then the system is exponentially stable, if $\alpha < 0$ and $\beta = \alpha + r > 0$, ($r > 0$), then the system is qualitatively exponentially stable.

If $\beta = \alpha + r < 0$, ($r > 0$), then the system qualitatively exponentially unstable.

To make the system (7) with the values of the parameter $q(t, x(t))$ of the limited bounded domain D_q meet the quality estimations (8) and (9) with the values of α and β , it is enough that a symmetric positively defined matrix P size $n \times n$ exists and with all values of $q \in D_q$ satisfies the inequality

$$\tilde{F}^T(q)P\tilde{F}(q) - r^2P \leq 0 \quad (10)$$

where $\tilde{F}(q) = F(q) + \beta I$, $\lambda = \beta + r$. With the setting we have $p = \sqrt{\frac{C_2}{C_1}}$, where C_2 and C_1 are maximum and minimum of the matrix P eigenvalue.

Checking of (10) is to find the maximum eigenvalue $\mu + (q)$ of the matrix $\tilde{F}^T(q)P\tilde{F}(q)P^{-1}$ and verification of (6) for all $q \in D_q$. Thus, to verify the conditions of the provisions for discrete and continuous systems we use the same software. Note that the results of works say that the condition (4) guarantees the position of all the eigenvalues $\lambda_i(q)$ ($i = 1, 2, \dots, n$) of the matrix $F(q)$ for all values of $q \in D_q$ within a circle with radius r centered at (β, j_0) of the complex plane, and condition (10) is within a circle of radius r centered at $(-\beta, j_0)$ of the complex plane. In other words, changing $q \in D_q$ the trajectory of discrete or continuous system roots are limited by above said areas of the complex plane.

Construction of the Ellipsoidal Estimates

We assume that the matrix $F(q)$ of systems (1) and (7) can be represented as

$$F(q) = F_0 - \sum_{i=1}^k B_i q_i^T \quad (11)$$

where F_0 is square matrix of size $n \times n$ with constant elements, B_i is matrix size $n \times 1$, q_i is n -dimensional vectors, which are functions of time and of current value of the state vector ($q_i = q_i(m, x(m))$).

It is thought out that the set B_i , $i = 1, 2, \dots, k$ ($1 \leq k \leq n$) form a linearly independent system of vectors. Introduce Lyapunov function

$$V(x) = x^T P x \quad (12)$$

given by the quadratic form with a positive definite symmetric matrix P of $n \times n$ size. First, consider the case when the matrix of a closed system is presented in the form (11) with $k = 1$. Let the matrix F_0 be such that the solution of the Lyapunov equation

$$\bar{F}_0^T P_0 \bar{F}_0 - r_0^2 P_0 = -Q_0 \quad (13)$$

with a positively defined matrix $Q_0 > 0$ of $n \times n$ size,

$$\bar{F}_0 = F_0 - \beta I,$$

$$\beta > 0,$$

$$0 < \beta + r_0 < 1,$$

is positive defined with $P_0 > 0$.

Then, in order to allow the system (1) meet the quality estimations (2), (3) with the parameters β and $r \geq r_0$, it is enough, that for each m and $x(m)$ values of q belong to the field D_q , limited by the surface of the ellipsoid

$$(q - q_N)^T D^{-1} (q - q_N) = (B^T P B)^{-1} \quad (14)$$

where

$$D = r^2 P - \bar{F}_0^T P \bar{F}_0 + q_N B^T P B q_N^T; \quad (15)$$

$$q_N = (B^T P_0 \bar{F}_0) \quad (16)$$

and the matrix P has the form

$$P = P_0 + \Delta P. \quad (17)$$

And ΔP at least positive semi-defined symmetrical matrix $n \times n$ size, which meet two conditions

$$D > 0; \quad (18)$$

$$B^T P = B^T P_0. \quad (19)$$

Let the matrix $P_0 > 0$ is a solution of the Lyapunov equation (13), and the system B_i , $e_i = 1, 2, \dots, k$ ($1 \leq k \leq n$) forms a system of linearly independent mutually orthogonal vectors in the sense of

$$B_i^T P_0 B_j = 0. \quad (20)$$

Then, in order to allow for the system (1) to meet the quality estimations (2), (3) with the parameters β and $r \geq r_0$, it is enough that for each m and $x(m)$ values of the vectors q_i belong to areas D_{q_i} , limited by the surface of the ellipsoid

$$(q_i - q_{iN})^T D_i^{-1} (q_i - q_{iN}) = (B_i^T P_0 B_i)^{-1} \quad (21)$$

where D_i are positively defined symmetric matrixes, such as

$$D = \sum_{i=1}^k D_i; \quad (22)$$

$$D = r^2 P_0 = \bar{F}_0^T P \bar{F}_0 + \sum_{i=1}^k q_{iN} B_i^T P_0 B_i q_{iN}^T; \quad (23)$$

$$q_{iN} = (B_{i0}^T P_0 \bar{F}_0). \quad (24)$$

For continuous systems with the equation of motion (7) with respect to the quality estimates (8) and (9) we have the same assertion of Theorem 3 and the corollary to it, if in (13)—(16) and (21)—(24) to replace matrix $\bar{F}$ with $\tilde{F} = F + \beta I$ matrix under the same limits on the values of β and r , as in (8) and (9).

Note that the values q_N (16) (q_{Ni} (24)), determine the minimum value of the function

$$V(x(m+1) = \beta x(m)) = x^T \bar{F}^T(q) P \bar{F}(q) x \quad (25)$$

i. e. when

$$\frac{\partial V(x(m+1) - \beta x(m))}{\partial q_{iN}} = 0$$

on the trajectories of the system (1), if the matrix $F(q)$ has the representation (11). In other words, the values $q_{Ni} \in D_{q_i}$ ask such vectors' changing values of parameters for which there is a minimum sensitivity

on the trajectories of the system. Note also that in the received results it is assumed that the functions $q_i(m, x)$ are of such type that for all values of $x \in R_n$ values of the vector of varying parameters belong to $D_{q_i}(q_i(m)) \in D_{q_i}$ area. If these conditions are not valid and q_i are functions only of the state vector ($q_i(m, x) = q_i(x)$), then equation (21) give the surfaces in the state space, limiting the area of D_{ix} , in which for $x \in D_{ix}$ run restrictions for changing parameters $q_i(x)$. Let these areas are simply connected and include the origin of coordinates. Then, for any initial condition $x(0)$ from the state space bounded by a surface

$$x^T P x = d^2$$

where

$$d^2 = \max x^T P x$$

for $x \in D_x$ trajectories of the system (1) or (7) we have the appropriate evaluation of quality (2), (3) or (8), (9).

Conclusion

Developed theoretical concepts come from the need to solve practical problems of analysis and motion control of moving objects [14, 15], especially in the automation of the most complicated modes of aircraft landing on mobile and immobile base, spatial tracking system during capturing and auto-tracking mode, the trajectory movements in robotic complexes, the management of chemical and other types of processes subjected to increased risk of accidents.

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