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Synthesis of Stabilization Laws of a Single-Airscrew Helicopter's Lateral Motion for Lack of Information about its Lateral Speed: Analytical Solution

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The problem of stabilization law synthesis of a single-airscrew helicopter's lateral motion for lack of information about the lateral speed of its motion is analytically solved. The solution is based on the method of output vector control synthesis, which provides a specified spectrum of a dynamic system motion to be used as the basis of an especially designed multilevel decomposition of the system model in state space. Numerical simulation data that confirm the results are presented.

Keywords: *linear MIMO-system, output vector control synthesis, modal synthesis*

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1. Introduction and problem statement

In practice, the synthesis of a single-airscrew helicopter (SH) control laws the approach of division of SH spatial motion onto isolated longitudinal and transversal motions is accepted [1]. In this case, in accordance with [1], the control object in the side channel can be considered as an interrelated (i.e. roll-yawing) motion of SH, which in the "input-state" form, is:

$$\dot{\mathrm{x}}(t) = \boldsymbol{A}\mathrm{x}(t) + \boldsymbol{B}\mathrm{u}(t), \tag{1}$$

$$\mathbf{x} = \begin{pmatrix} \Delta V_z \\ \Delta\omega_x \\ \Delta\omega_y \\ \Delta\gamma \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \Delta u_z \\ \Delta u_{\text{рв}} \end{pmatrix},$$

having the following matrices of coefficients:

$$\boldsymbol{A} = \begin{pmatrix} a_{V_z}^{V_z} & a_{V_z}^{\omega_x} & a_{V_z}^{\omega_y} & a_{V_z}^{\gamma} \\ a_{\omega_x}^{V_z} & a_{\omega_x}^{\omega_x} & a_{\omega_x}^{\omega_y} & 0 \\ a_{\omega_y}^{V_z} & a_{\omega_y}^{\omega_x} & a_{\omega_y}^{\omega_y} & 0 \\ 0 & 1 & a_{\gamma}^{\omega_y} & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} b_{V_z}^{u_z} & b_{V_z}^{u_{\text{рв}}} \\ b_{\omega_x}^{u_z} & b_{\omega_x}^{u_{\text{рв}}} \\ b_{\omega_y}^{u_z} & b_{\omega_y}^{u_{\text{рв}}} \\ 0 & 0 \end{pmatrix}.$$

Here, the elements of the matrices

$$a_{V_z}^{V_z},\ a_{V_z}^{\omega_x},\ a_{V_z}^{\omega_y},\ a_{V_z}^{\gamma},\ a_{\omega_x}^{V_z},\ a_{\omega_x}^{\omega_x},\ a_{\omega_x}^{\omega_y},\ a_{\omega_y}^{V_z},$$

$$a_{\omega_y}^{\omega_x},\ a_{\omega_y}^{\omega_y},\ a_{\gamma}^{\omega_y},\ b_{V_z}^{u_z},\ b_{V_z}^{u_{\text{рв}}},\ b_{\omega_x}^{u_z},\ b_{\omega_x}^{u_{\text{рв}}},\ b_{\omega_y}^{u_z},\ b_{\omega_y}^{u_{\text{рв}}}$$

are piecewise constant values (i.e. linearization coefficients: [1]). The variables that correspond to the vectors of state and entry (of control) have the following meanings: ΔV_z — deviation from specified value of the lateral speed; $\Delta\omega_x$ — deviation from specified value of the roll angular velocity; $\Delta\omega_y$ — deviation from specified value of the yawing angular velocity; $\Delta\gamma$ — deviation from specified value of the angle of roll; Δu_z — deviation angle of a main rotor's cone in the transverse direction; and $\Delta u_{\text{рв}}$ — pitch of a steering propeller.

We use the following notation:

$$a_{11} = a_{V_z}^{V_z}, a_{12} = a_{V_z}^{\omega_x}, a_{13} = a_{V_z}^{\omega_y}, a_{14} = a_{V_z}^{\gamma},$$

$$a_{21} = a_{\omega_x}^{V_z}, a_{22} = a_{\omega_x}^{\omega_x}, a_{23} = a_{\omega_x}^{\omega_y},$$

$$a_{31} = a_{\omega_y}^{V_z}, a_{32} = a_{\omega_y}^{\omega_x}, a_{33} = a_{\omega_y}^{\omega_y}, a_{43} = a_{\gamma}^{\omega_y},$$

$$b_{11} = b_{V_z}^{u_z}, b_{12} = b_{V_z}^{u_{\text{рв}}}, b_{21} = b_{\omega_x}^{u_z}, b_{22} = b_{\omega_x}^{u_{\text{рв}}},$$

$$b_{31} = b_{\omega_y}^{u_z}, b_{32} = b_{\omega_y}^{u_{\text{рв}}},$$

then the control object (1) as a Multi Inputs Multi Outputs (MIMO) system of the "input-state" type can be written in more detail in the following way:

$$\begin{pmatrix} \Delta\dot{V}_z \\ \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \Delta\dot{\gamma} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 1 & a_{43} & 0 \end{pmatrix} \begin{pmatrix} \Delta V_z \\ \Delta\omega_x \\ \Delta\omega_y \\ \Delta\gamma \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta u_z \\ \Delta u_{\text{рв}} \end{pmatrix}. \tag{2}$$

Hereafter it will be considered that information about change of speed ΔV_z as a result of direct or indirect measurements is not available.

Taking into consideration the assumptions made, the vector differential equation (2) can be written in the form of a dynamic MIMO-system of the "input — state — output" type:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \boldsymbol{C}\mathbf{x}(t), \tag{3}$$

where the matrices with real elements (i.e. those specified over the field of real numbers $\mathbb{R}$) are equal to:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 1 & a_{43} & 0 \end{pmatrix} \in \mathbb{R}^{n\times n},\ n = 4, \tag{4}$$

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times r},\ n = 4,\ r = 2, \tag{5}$$

$$\boldsymbol{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{m\times n},\ n = 4,\ m = 3. \tag{6}$$

If, as a control law (3), to suggest the expression of the following form:

$$\mathbf{u}(t) = \boldsymbol{F}\mathbf{y}(t) = \boldsymbol{FC}\mathbf{x}(t), \tag{7}$$

where $\boldsymbol{F} \in \mathbb{R}^{r\times m}$ is a matrix of the output controller, then, in accordance with [2] for the system under consideration (3)—(7), a case of the dynamic MIMO-system output vector control (i.e. output control) will take place.

It is required (with the help of the control law (7)) to provide the specified motion spectrum for the controlled system (3).

It should be noted that the output control of the spectrum of a dynamic system is a classical problem in control theory; however, judging by multiple published works, in which different mathematical approaches are used (e.g. [3—7]), no complete solution of this problem is presently available.

Hereafter we assume that matrix $\boldsymbol{B} \in \mathbb{R}^{4\times 2}$ (5) has a full rank (rank$\boldsymbol{B}$ = 2 in this case), or that its equivalent matrix $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}$ is invertible (i.e. $\det(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}) \neq 0$). From a physical standpoint, this easily performed requirement means a linear independence of input control signals.

Let us now specify a notion of the spectrum considered here. It will be understood as a set of matrix $\boldsymbol{A}$ eigenvalues. In this case $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ (4) and a set of eigenvalues can be presented in the following way:

$$\operatorname{eig}(\boldsymbol{A}) = \{\lambda_i \in \mathbb{C}:\ \det(\lambda_i \boldsymbol{I}_4 - \boldsymbol{A}) = 0,\ i = 1,\ldots,4\}.$$

Here, $\boldsymbol{I}_4$ — identity matrix of size 4×4, $\mathbb{C}$ — the set of complex numbers (complex plane).

Let Λ be the given spectrum of matrix of system (3) with a close-loop control (7), then it is possible to determine a spectrum of the close-loop system as a set of the matrix $\boldsymbol{A} + \boldsymbol{BFC}$ eigenvalues, that is

$$\Lambda = \{\hat{\lambda}_1,\ \hat{\lambda}_2, \hat{\lambda}_3,\ \hat{\lambda}_4\}. \tag{8}$$

Thus, it is required to determine (i.e. synthesize) explicitly the controller matrix $\boldsymbol{F} \in \mathbb{R}^{2\times 3}$ (7), such that the equality

$$\Lambda = \operatorname{eig}(\boldsymbol{A} + \boldsymbol{BFC})$$

should be satisfied exactly.

The additional (methodological) complexity of this problem is a necessity for obtaining a solution in explicit analytical form, since $\boldsymbol{A}$, $\boldsymbol{B}$ matrices, at best, as per [1], have a piecewise constant form. We emphasize that we know nothing about any alternative approach that allows the analytical solution of this problem to be obtained.

2. Decomposition of a dynamic system. As a first step of the given problem solution we will consider the multilevel decomposition of the SH model suggested in [8—10].

Since in this case the inequality $m \geqslant r$ (i.e. the number of system's outputs is greater than the number of its inputs) is implemented, then, in general, not taking into consideration specific numerical values for m and r, we consider the multilevel decomposition of system (3) of the following form:

— zero (initial) decomposition level

$$\boldsymbol{A}_0 = \boldsymbol{A},\ \boldsymbol{B}_0 = \boldsymbol{B},\ \boldsymbol{C}_0 = \boldsymbol{C}, \tag{9}$$

— first decomposition level

$$\boldsymbol{A}_1 = \boldsymbol{B}_0^{\perp}\boldsymbol{A}_0\boldsymbol{B}_0^{\perp\mathrm{T}},\ \boldsymbol{B}_1 = \boldsymbol{B}_0^{\perp}\boldsymbol{A}_0\boldsymbol{B}_0, \boldsymbol{C}_1 = \boldsymbol{C}_0\boldsymbol{A}_0\boldsymbol{B}_0^{\perp\mathrm{T}}, \tag{10}$$

— kth decomposition level ($1 < k < M$)

$$\begin{aligned} &\boldsymbol{A}_k = \boldsymbol{B}_{k-1}^{\perp}\boldsymbol{A}_{k-1}\boldsymbol{B}_{k-1}^{\perp\mathrm{T}},\ \boldsymbol{B}_k = \boldsymbol{B}_{k-1}^{\perp}\boldsymbol{A}_{k-1}\boldsymbol{B}_{k-1}, \\ &\boldsymbol{C}_k = \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1}\boldsymbol{B}_{k-1}^{\perp\mathrm{T}}, \end{aligned} \tag{11}$$

— Mth (final) decomposition level (here: M = = ceil(n/r), where ceil(*) — is the operation of rounding the number "*" upwards)

$$\begin{aligned} &\boldsymbol{A}_M = \boldsymbol{B}_{M-1}^{\perp}\boldsymbol{A}_{M-1}\boldsymbol{B}_{M-1}^{\perp+},\ \boldsymbol{B}_M = \boldsymbol{B}_{M-1}^{\perp}\boldsymbol{A}_{M-1}\boldsymbol{B}_{M-1}, \\ &\boldsymbol{C}_M = \boldsymbol{C}_{M-1}\boldsymbol{A}_{M-1}\boldsymbol{B}_{M-1}^{\perp+}. \end{aligned} \tag{12}$$

Equations (9)—(12) for a set of indices $k = \overline{0, M}$ involve the matrices with the following properties:

$$\left(\boldsymbol{B}_k \mid \boldsymbol{B}_k^{\perp\mathrm{T}}\right)^{-1} = \begin{pmatrix} \boldsymbol{B}_k^{+} \\ \hline \boldsymbol{B}_k^{\perp} \end{pmatrix},\ \boldsymbol{B}_k^{\perp}\boldsymbol{B}_k = 0,\ \boldsymbol{B}_k^{+}\boldsymbol{B}_k = \boldsymbol{I}_r, \tag{13}$$

$$\begin{pmatrix} \boldsymbol{C}_k \\ \hline \boldsymbol{C}_k^{\perp} \end{pmatrix}^{-1} = \left(\boldsymbol{C}_k^{+} \mid \boldsymbol{C}_k^{\perp\mathrm{T}}\right),\ \boldsymbol{C}_k^{\perp}\boldsymbol{C}_k^{\mathrm{T}} = 0,\ \boldsymbol{C}_k\boldsymbol{C}_k^{+} = \boldsymbol{I}_m, \tag{14}$$

where the superscript "T" denotes the transposition operation, the superscript " ⊥" denotes semi-orthogonal annihilators (divisors of zero), and the superscript "+" denotes the Moore-Penrose pseudoinverse matrices [8—10].

Also, we consider the recurrence formulae to obtain the required controller in (7), written down in reverse order:

— M-th (final) decomposition level

$$F_M = \left(\Phi_M B_M^{+} - B_M^{+} A_M\right) C_M^{+}, \qquad (15)$$

— k-th decomposition level ($1 < k < M$)

$$F_k = \left(\Phi_k B_k^{-} - B_k^{-} A_k\right) C_k^{+}, \quad B_k^{-} = B_k^{+} - F_{k-1} B_k^{\perp}, \qquad (16)$$

— first decomposition level

$$F_1 = \left(\Phi_1 B_1^{-} - B_1^{-} A_1\right) C_1^{+}, \quad B_1^{-} = B_1^{+} - F_2 B_1^{\perp}, \qquad (17)$$

— zero (initial) decomposition level

$$F_0 = \left(\Phi_0 B_0^{-} - B_0^{-} A_0\right) C_0^{+}, \quad B_0^{-} = B_0^{+} - F_1 B_0^{\perp}. \qquad (18)$$

Here Φ_i ($i = \overline{0, M}$) are certain specified matrices, which will be determined in the next section.

The multilevel decomposition procedure considered is then implemented.

3. Algorithm for synthesis of the MIMO-system output control

The following statement that has been proven in [11] is true

Theorem 1. *Let $m \geqslant r$, and the following matrices exist and are pairwise completely controllable:*

$$G_M^{\mathrm{T}} = B_M^{+} A_M C_M^{\perp} \left(B_M^{+} C_M^{\perp}\right)^{+}, \quad H_M^{\mathrm{T}} = \left(B_M^{+} C_M^{\perp}\right)^{\perp}, \qquad (19)$$

$$G_k^{\mathrm{T}} = B_k^{-} A_k C_k^{\perp} \left(B_k^{-} C_k^{\perp}\right)^{+}, \quad H_k^{\mathrm{T}} = \left(B_k^{-} C_k^{\perp}\right)^{\perp}, \qquad (20)$$

$$G_1^{\mathrm{T}} = B_1^{-} A_1 C_1^{\perp} \left(B_1^{-} C_1^{\perp}\right)^{+}, \quad H_1^{\mathrm{T}} = \left(B_1^{-} C_1^{\perp}\right)^{\perp}, \qquad (21)$$

$$G_0^{\mathrm{T}} = B_0^{-} A_0 C_0^{\perp} \left(B_0^{-} C_0^{\perp}\right)^{+}, \quad H_0^{\mathrm{T}} = \left(B_0^{-} C_0^{\perp}\right)^{\perp}. \qquad (22)$$

Then, there exists a nonempty set of matrices K_i, $i = \overline{0, M}$, such that

$$\Phi_i = G_i + K_i^{\mathrm{T}} H_i = \left(B_i^{-} A_i C_i^{\perp}\right)\left(B_i^{-} C_i^{\perp}\right)^{+} + K_i^{\mathrm{T}} \left(B_i^{-} C_i^{\perp}\right)^{\perp}, \qquad (23)$$

and (19)— (22) *satisfy the equalities of spectra*

$$\operatorname{eig}\left(A_M + B_M F_M C_M\right) = \operatorname{eig}(\Phi_M), \qquad (24)$$

$$\operatorname{eig}\left(A_k + B_k F_k C_k\right) = \bigcup_{i=k-1}^{M} \operatorname{eig}(\Phi_i), \qquad (25)$$

$$\operatorname{eig}\left(A_1 + B_1 F_1 C_1\right) = \bigcup_{i=1}^{M} \operatorname{eig}(\Phi_i), \qquad (26)$$

$$\operatorname{eig}\left(A_0 + B_0 F_0 C_0\right) = \operatorname{eig}\left(A + BFC\right) = \bigcup_{i=1}^{M+1} \operatorname{eig}(\Phi_i) = \Lambda. \qquad (27)$$

The condition $m \geqslant r$ in Theorem 1 is not restrictive; it is introduced to indicate that, in the present case, F matrix from (7) is conventionally considered as a matrix of controller (i.e. the number of inputs is less than the number of outputs), and not as a matrix of state observer (i.e. the number of inputs is greater than the number of outputs).

For the case $m \leqslant r$. Theorem 1 has a dual formulation, and matrix F is replaced with the observer matrix L.

Theorem 2. *Let $m \leqslant r$, N = ceil(n/m), and the following decomposition of system (3) hold* ($1 < k < N$):

$$A_0 = A, \quad B_0 = B, \quad C_0 = C,$$

$$A_1 = C_0^{\perp} A_0 C_0^{\perp\mathrm{T}}, \quad B_1 = C_0^{\perp} A_0 C_0, \quad C_1 = C_0 A_0 C_0^{\perp\mathrm{T}},$$

$$A_k = C_{k-1}^{\perp} A_{k-1} C_{k-1}^{\perp\mathrm{T}}, \quad B_k = C_{k-1}^{\perp} A_{k-1} C_{k-1}, \quad C_k = C_{k-1} A_{k-1} C_{k-1}^{\perp\mathrm{T}},$$

$$A_N = C_{N-1}^{\perp} A_{N-1} C_{N-1}^{\perp\mathrm{T}}, \quad B_N = C_{N-1}^{\perp} A_{N-1} C_{N-1}, \quad C_N = C_{N-1} A_{N-1} C_{N-1}^{\perp\mathrm{T}},$$

moreover, the following matrices exist and are pairwise completely controllable:

$$G_N = \left(B_N^{\perp} C_N^{+}\right)^{+} B_N^{\perp} A_N C_N^{+}, \quad H_N = \left(B_N^{\perp} C_N^{+}\right)^{\perp},$$

$$G_k = \left(B_k^{\perp} C_k^{+}\right)^{+} B_k^{\perp} A_k C_k^{+}, \quad H_k = \left(B_k^{\perp} C_k^{+}\right)^{\perp},$$

$$G_1 = \left(B_1^{\perp} C_1^{+}\right)^{+} B_1^{\perp} A_1 C_1^{+}, \quad H_1 = \left(B_1^{\perp} C_1^{+}\right)^{\perp},$$

$$G_0 = \left(B_0^{\perp} C_0^{+}\right)^{+} B_0^{\perp} A_0 C_0^{+}, \quad H_0 = \left(B_0^{\perp} C_0^{+}\right)^{\perp}.$$

Then, there exists a nonempty set of matrices L_i, $i = \overline{0, N}$, such that

$$\Psi_i = G_i + H_i L_i^{\mathrm{T}} = \left(B_i^{\perp} C_i^{+}\right)^{+} B_i^{\perp} A_i C_i^{+} + \left(B_i^{\perp} C_i^{+}\right)^{\perp} L_i^{\mathrm{T}},$$

and, for

$$\boldsymbol{F}_N = \boldsymbol{B}_M^+\left(\boldsymbol{C}_M^+\Psi_M - \boldsymbol{A}_M\boldsymbol{C}_M^+\right),$$

$$\boldsymbol{F}_k = \boldsymbol{B}_k^+\left(\boldsymbol{C}_k^-\Psi_k - \boldsymbol{A}_k\boldsymbol{C}_k^-\right), \quad \boldsymbol{C}_k^- = \boldsymbol{C}_k^+ - \boldsymbol{C}_k^{\perp\mathrm{T}}\boldsymbol{F}_{k-1},$$

$$\boldsymbol{F}_1 = \boldsymbol{B}_1^+\left(\boldsymbol{C}_1^-\Psi_1 - \boldsymbol{A}_1\boldsymbol{C}_1^-\right), \quad \boldsymbol{C}_1^- = \boldsymbol{C}_1^+ - \boldsymbol{C}_1^{\perp\mathrm{T}}\boldsymbol{F}_2,$$

$$\boldsymbol{F}_0 = \boldsymbol{B}_0^+\left(\boldsymbol{C}_0^-\Psi_0 - \boldsymbol{A}_0\boldsymbol{C}_0^-\right), \quad \boldsymbol{C}_0^- = \boldsymbol{C}_0^+ - \boldsymbol{C}_0^{\perp\mathrm{T}}\boldsymbol{F}_1,$$

it holds that

$$\mathrm{eig}\left(\boldsymbol{A}_N + \boldsymbol{B}_N\boldsymbol{F}_N\boldsymbol{C}_N\right) = \mathrm{eig}(\Psi_N),$$

$$\mathrm{eig}\left(\boldsymbol{A}_k + \boldsymbol{B}_k\boldsymbol{F}_k\boldsymbol{C}_k\right) = \bigcup_{i=k-1}^{N} \mathrm{eig}(\Psi_i),$$

$$\mathrm{eig}\left(\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{F}_1\boldsymbol{C}_1\right) = \bigcup_{i=1}^{N} \mathrm{eig}(\Psi_i),$$

$$\mathrm{eig}\left(\boldsymbol{A}_0 + \boldsymbol{B}_0\boldsymbol{F}_0\boldsymbol{C}_0\right) = \mathrm{eig}\left(\boldsymbol{A} + \boldsymbol{BFC}\right) = \bigcup_{i=1}^{N+1} \mathrm{eig}(\Psi_i) = \Lambda.$$

As in the algorithms described in [8—10], only semiorthogonal and pseudoinverse matrices are used in the transformations, which at least do not reduce the condition number of the equations.

This approach does not impose restrictions in the form of the differentiation between the algebraic and geometric multiplicities of the elements of the spectrum to be assigned; there are also no restrictions on the size of the problem [8—10]. This is confirmed by extensive simulation, which shows a high relative accuracy of spectrum control and the practical absence of restrictions on the size of system (3).

4. Analytical synthesis of aircraft's lateral motion control

In accordance with the problem statement, it is required to find explicitly a formula of controller $\boldsymbol{F}$ in the control law that can be expressed in this case as:

$$\begin{pmatrix} \Delta u_z(t) \\ \Delta u_{\text{рв}}(t) \end{pmatrix} = \boldsymbol{FC}\begin{pmatrix} \Delta V_z \\ \Delta\omega_x \\ \Delta\omega_y \\ \Delta\gamma \end{pmatrix} = \boldsymbol{F}\begin{pmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\gamma \end{pmatrix}, \qquad (28)$$

and provides for the close-loop system "HS + control system" of a specified spectrum (8).

We perform for the system (3) with matrices (4)—(6) the multilevel decomposition described in Section 2, which has in this case two decomposition levels ($M = 1$): zero level (9) and first level (10). Therefore, we will have

$$\boldsymbol{B}_0^{\perp} = \begin{pmatrix} \dfrac{b_{21}b_{32} - b_{22}b_{31}}{b_{11}b_{22} - b_{12}b_{21}} & -\dfrac{b_{11}b_{32} - b_{12}b_{31}}{b_{11}b_{22} - b_{12}b_{21}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{B}_0^{\perp+} = \begin{pmatrix} b_{11}^+ & 0 \\ b_{21}^+ & 0 \\ b_{31}^+ & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{A}_1 = \begin{pmatrix} a_{*11} & a_{*12} \\ a_{*21} & 0 \end{pmatrix}, \quad \boldsymbol{B}_1 = \begin{pmatrix} b_{*11} & b_{*12} \\ b_{*21} & b_{*22} \end{pmatrix}, \quad \boldsymbol{C}_1 = \begin{pmatrix} c_{*11} & 0 \\ c_{*21} & 0 \\ c_{*31} & 0 \end{pmatrix},$$

where for the compactness of a record we use the following symbols:

$$b_{11}^+ = (b_{11}b_{22} - b_{12}b_{21})(b_{21}b_{32} - b_{22}b_{31})/b^{+*},$$
$$b_{21}^+ = -(b_{11}b_{22} - b_{12}b_{21})(b_{11}b_{32} - b_{12}b_{31})/b^{+*},$$
$$b_{31}^+ = (b_{11}b_{22} - b_{12}b_{21})^2/b^{+*},$$
$$b^{+*} = b_{11}^2b_{22}^2 + b_{11}^2b_{32}^2 - 2b_{11}b_{12}b_{21}b_{22} -$$
$$-2b_{11}b_{12}b_{31}b_{32} + b_{12}^2b_{21}^2 + b_{12}^2b_{31}^2 +$$
$$+b_{21}^2b_{32}^2 - 2b_{21}b_{22}b_{31}b_{32} + b_{22}^2b_{31}^2,$$
$$a_{*11} = (b_{21}b_{32} - b_{22}b_{31})(a_{11}b_{22}b_{11}^+ - a_{21}b_{12}b_{11}^+ +$$
$$+ a_{12}b_{22}b_{21}^+ - a_{22}b_{12}b_{21}^+ + a_{13}b_{22}b_{31}^+ -$$
$$-a_{23}b_{12}b_{31}^+)/(b_{22}(b_{11}b_{22} - b_{12}b_{21})) -$$
$$-(a_{21}b_{32}b_{11}^+ - a_{31}b_{22}b_{11}^+ + a_{22}b_{32}b_{21}^+ - a_{32}b_{22}b_{21}^+ +$$
$$+a_{23}b_{32}b_{31}^+ - a_{33}b_{22}b_{31}^+)/b_{22}, a_{*12} =$$
$$= a_{14}(b_{21}b_{32} - b_{22}b_{31})/(b_{11}b_{22} - b_{12}b_{21}),$$
$$a_{*21} = b_{21}^+ + a_{43}b_{31}^+, b_{*11} = (a_{31}b_{11}^2 - a_{13}b_{31}^2 -$$
$$-a_{11}b_{11}b_{31} - a_{12}b_{21}b_{31} + a_{32}b_{11}b_{21} + a_{33}b_{11}b_{31})/b_{11} -$$
$$-(b_{11}b_{32} - b_{12}b_{31})(a_{21}b_{11}^2 - a_{12}b_{21}^2 - a_{11}b_{11}b_{21} +$$
$$+ a_{22}b_{11}b_{21} - a_{13}b_{21}b_{31} + a_{23}b_{11}b_{31})/(b_{11}(b_{11}b_{22} - b_{12}b_{21})),$$
$$b_{*12} = (a_{32}b_{22}^2 - a_{23}b_{32}^2 - a_{21}b_{12}b_{32} + a_{31}b_{12}b_{22} -$$
$$-a_{22}b_{22}b_{32} + a_{33}b_{22}b_{32})/b_{22} - (a_{21}b_{12}^2 - a_{12}b_{22}^2 -$$
$$-a_{11}b_{12}b_{22} + a_{22}b_{12}b_{22} - a_{13}b_{32}b_{22} + a_{23}b_{12}b_{32})\times$$
$$\times (b_{21}b_{32} - b_{22}b_{31})/(b_{22}(b_{11}b_{22} - b_{12}b_{21})),$$
$$b_{*21} = b_{21} + a_{43}b_{31}, b_{*22} = b_{22} + a_{43}b_{32},$$
$$c_{*11} = a_{21}b_{11}^+ + a_{22}b_{21}^+ + a_{23}b_{31}^+,$$
$$c_{*21} = a_{31}b_{11}^+ + a_{32}b_{21}^+ + a_{33}b_{31}^+,$$
$$c_{*31} = b_{21}^+ + a_{43}b_{31}^+.$$

To check the controllability conditions in Theorem 1, we calculate the matrices:

$$\boldsymbol{C}_0^{\perp\mathrm{T}} = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B}_0^+ = \begin{pmatrix} b_{11}^{0+} & b_{12}^{0+} & b_{13}^{0+} & 0 \\ b_{21}^{0+} & b_{22}^{0+} & b_{23}^{0+} & 0 \end{pmatrix},$$

$$H_0 = \left(B_0^+ C_0^\perp\right)^{\perp T} = \begin{pmatrix} \frac{-b_{21}^{0+}}{b_{11}^{0+}} \\ 1 \end{pmatrix},$$

$$G_0 = \begin{pmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{pmatrix},$$

$$C_1^{\perp T} = (0 \quad 1), \quad B_1^+ = \begin{pmatrix} b_{11}^{1+} & b_{12}^{1+} \\ b_{21}^{1+} & b_{22}^{1+} \end{pmatrix},$$

$$H_1 = \left(B_1^+ C_1^{\perp T}\right)^{\perp T} = \begin{pmatrix} \frac{b_{22}^{1+}}{b_{12}^{1+}} \\ 1 \end{pmatrix}, \quad H_1^+ = \left(b_{11}^{p1} \quad b_{12}^{p1}\right),$$

$$G_1 = \left(B_1^+ C_1^{\perp T}\right)^{+T} \left(B_1^+ A_1 C_1^{\perp T}\right)^T = \begin{pmatrix} a_{11}^{a1} & a_{12}^{a1} \\ a_{21}^{a1} & a_{22}^{a1} \end{pmatrix},$$

where, as before, the following notation is used:

$$b_{11}^{0+} = -\frac{b_{12}(b_{21}b_{22} + b_{31}b_{32}) - b_{11}(b_{22}^2 + b_{32}^2)}{b^{+*}},$$

$$b_{12}^{0+} = -\frac{b_{22}(b_{11}b_{12} + b_{31}b_{32}) - b_{21}(b_{12}^2 + b_{32}^2)}{b^{+*}},$$

$$b_{13}^{0+} = -\frac{b_{32}(b_{11}b_{12} + b_{21}b_{22}) - b_{31}(b_{12}^2 + b_{22}^2)}{b^{+*}},$$

$$b_{21}^{0+} = -\frac{b_{11}(b_{21}b_{22} + b_{31}b_{32}) - b_{12}(b_{21}^2 + b_{31}^2)}{b^{+*}},$$

$$b_{22}^{0+} = -\frac{b_{21}(b_{11}b_{12} + b_{31}b_{32}) - b_{22}(b_{11}^2 + b_{31}^2)}{b^{+*}},$$

$$b_{23}^{0+} = -\frac{b_{31}(b_{11}b_{12} + b_{21}b_{22}) - b_{32}(b_{11}^2 + b_{21}^2)}{b^{+*}},$$

$$a_{11}^0 = b_{11}^{0+} b_{*1}^{0+}, a_{12}^0 = b_{11}^{0+} b_{*2}^{0+}, a_{21}^0 = b_{12}^{0+} b_{*1}^{0+}, a_{22}^0 = b_{12}^{0+} b_{*2}^{0+},$$

$$b_{*1}^{0+} = a_{11} b_{11}^{0+} + a_{21} b_{12}^{0+} + a_{31} b_{13}^{0+},$$

$$b_{*2}^{0+} = a_{11} b_{21}^{0+} + a_{21} b_{22}^{0+} + a_{31} b_{23}^{0+},$$

$$b_{11}^{1+} = b_{*22}/(b_{*11}b_{*22} - b_{*12}b_{*21}),$$

$$b_{12}^{1+} = -b_{*12}/(b_{*11}b_{*22} - b_{*12}b_{*21}),$$

$$b_{21}^{1+} = -b_{*21}/(b_{*11}b_{*22} - b_{*12}b_{*21}),$$

$$b_{22}^{1+} = b_{*11}/(b_{*11}b_{*22} - b_{*12}b_{*21}),$$

$$b_{11}^{p1} = \frac{b_{12}^{1+}}{\left(b_{12}^{1+}\right)^2 + \left(b_{22}^{1+}\right)^2}, b_{12}^{p1} = \frac{b_{22}^{1+}}{\left(b_{12}^{1+}\right)^2 + \left(b_{22}^{1+}\right)^2},$$

$$a_{11}^{a1} = a_{*12} b_{11}^{1+} b_{11}^{p1}, a_{12}^{a1} = a_{*12} b_{21}^{1+} b_{11}^{p1},$$

$$a_{21}^{a1} = a_{*12} b_{11}^{1+} b_{12}^{p1}, a_{22}^{a1} = a_{*12} b_{21}^{1+} b_{12}^{p1}.$$

For the zero and first decomposition levels, we calculate the ranks of the following block matrices:

$$(H_0, \quad G_0 H_0), \; (H_1, \quad G_1 H_1),$$

as a result we will obtain:

$$\operatorname{rank}\left(H_0 \quad G_0 H_0\right) = \operatorname{rank}\left(H_1 \quad G_1 H_1\right) = 2;$$

this corresponds to the number of "independent" inputs $r = 2$. Therefore, each level of decomposition satisfies the control-lability condition in Theorem 1.

According to the form of controllers — we define a matrix whose eigenvalues will be assigned to the first decomposition level. With this purpose for matrices H_1, G_1 of the first decomposition level we will consider an additional sublevel, and calculate beforehand for this matrix $H_1^\perp$, which in this case is equal to

$$H_1^\perp = \begin{pmatrix} \frac{b_{12}^{1+}}{b_{22}^{1+}} & 1 \end{pmatrix}.$$

Next, using the expressions

$$(G_1)_1 = H_1^\perp G_1 H_1^{\perp T}, \; (H_1)_1 = H_1^\perp G_1 H_1,$$

we obtain

$$(H_1)_1 =$$
$$= a_{22}^{a1} - (b_{12}^{1+}(a_{21}^{a1} + a_{11}^{a1} b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} + a_{12}^{a1} b_{12}^{1+}/b_{22}^{1+},$$

$$(G_1)_1 = a_1^{aa} =$$
$$= a_{22}^{a1} + (b_{12}^{1+}(a_{21}^{a1} + a_{11}^{a1} b_{12}^{1+}/b_{22}^{1+}))/b_{22}^{1+} + a_{12}^{a1} b_{12}^{1+}/b_{22}^{1+}.$$

Whereupon the scalar value $(H_1)_1^+$ will be equal to

$$(H_1)_1^+ = \frac{1}{(H_1)_1}.$$

Let us now assign one of the eigenvalues as a scalar matrix

$$(\Phi_1)_1 = \hat{\lambda}_1 = s_{12}$$

and calculate the matrix of feedback coefficients for the additional sublevel of the first decomposition level. We obtain

$$k_1 = s_{11}/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1} b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} +$$
$$+ a_{12}^{a1} b_{12}^{1+}/b_{22}^{1+}) - (a_{22}^{a1} + (b_{12}^{1+}(a_{21}^{a1} + a_{11}^{a1} b_{12}^{1+}/b_{22}^{1+}))/b_{22}^{1+} +$$
$$+ a_{12}^{a1} b_{12}^{1+}/b_{22}^{1+})/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1} b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} +$$
$$+ a_{12}^{a1} b_{12}^{1+}/b_{22}^{1+}).$$

Next, according to equations — from Theorem 1 we calculate the matrix

$$(H_1)_0^- = (H_1)_0^+ - k_1 (H_1)_0^\perp = \left(b_{11}^{1m} \quad b_{12}^{1m}\right),$$

where

$$b_{11}^{1m} = -(b_{12}^{1+}(s_{11}/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+}) - (a_{22}^{a1} + (b_{12}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{22}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+})/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+})))/b_{22}^{1+} - b_{12}^{1+}b_{22}^{1+}/((b_{12}^{1+})^2 + (b_{22}^{1+})^2),$$

$$b_{12}^{1m} = (a_{22}^{a1} + (b_{12}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{22}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+})/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{12}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+}) - s_{11}/(a_{22}^{a1} - (b_{22}^{1+}(a_{21}^{a1} + a_{11}^{a1}b_{12}^{1+}/b_{22}^{1+}))/b_{22}^{1+}))/b_{12}^{1+} + a_{12}^{a1}b_{12}^{1+}/b_{22}^{1+}) + (b_{12}^{1+})^2/((b_{12}^{1+})^2 + (b_{22}^{1+})^2).$$

We then specify the matrix of eigenvalues of the zero sublevel of the first decomposition level by

$$(\Phi_1)_0 = \hat{\lambda}_2 = s_{12}.$$

Finally, we find matrix $\boldsymbol{k}_0$ by the rule

$$\boldsymbol{k}_0 = (\Phi_1)_0 (\boldsymbol{H}_1)_0^- - (\boldsymbol{H}_1)_0^- \boldsymbol{G}_1 = (k_{11} \quad k_{12}),$$

where

$$k_{11} = b_{11}^{1m}s_{12} - a_{21}^{a1}b_{21}^{1m} - a_{11}^{a1}b_{11}^{1m},$$
$$k_{11} = b_{21}^{1m}s_{12} - a_{22}^{a1}b_{21}^{1m} - a_{12}^{a1}b_{11}^{1m}.$$

As a result, we obtain, using equation (18), the matrix Φ_1, whose eigenvalues s_{12}, s_{12}, are ensured by the output controller for the model on the first decomposition level:

$$\Phi_1 = \begin{pmatrix} f_{11}^{i1} & f_{12}^{i1} \\ f_{21}^{i1} & f_{22}^{i1} \end{pmatrix}.$$

Here

$$f_{11}^{i1} = a_{11}^{a1} + (b_{22}^{1+}(a_{11}^{a1}b_{11}^{1m} + a_{21}^{a1}b_{21}^{1m} - b_{11}^{1m}s_{12})) / b_{12}^{1+},$$
$$f_{12}^{i1} = a_{21}^{a1} - a_{11}^{a1}b_{11}^{1m} - a_{21}^{a1}b_{21}^{1m} + b_{11}^{1m}s_{12},$$
$$f_{21}^{i1} = a_{12}^{a1} + (b_{22}^{1+}(a_{12}^{a1}b_{11}^{1m} + a_{22}^{a1}b_{21}^{1m} - b_{21}^{1m}s_{12})) / b_{12}^{1+},$$
$$f_{12}^{i1} = a_{22}^{a1} - a_{12}^{a1}b_{11}^{1m} - a_{22}^{a1}b_{21}^{1m} + b_{21}^{1m}s_{12}.$$

Based on equation (15), the first decomposition level yields the following formula for the controller

$$\boldsymbol{F}_1 = \left(\Phi_1 \boldsymbol{B}_1^+ - \boldsymbol{B}_1^+ \boldsymbol{A}_1\right)\boldsymbol{C}_1^+ = \begin{pmatrix} f_{11}^1 & f_{12}^1 & f_{13}^1 \\ f_{21}^1 & f_{22}^1 & f_{23}^1 \end{pmatrix},$$

where the elements are:

$$f_{11}^1 = -c_{11}^{1+}(a_{11}^{a1}b_{11}^{1+} + a_{21}^{a1}b_{12}^{1+} - b_{11}^{1+}f_{11}^{i1} - b_{21}^{1+}f_{12}^{i1}),$$
$$f_{12}^1 = -c_{12}^{1+}(a_{11}^{a1}b_{11}^{1+} + a_{21}^{a1}b_{12}^{1+} - b_{11}^{1+}f_{11}^{i1} - b_{21}^{1+}f_{12}^{i1}),$$
$$f_{13}^1 = -c_{13}^{1+}(a_{11}^{a1}b_{11}^{1+} + a_{21}^{a1}b_{12}^{1+} - b_{11}^{1+}f_{11}^{i1} - b_{21}^{1+}f_{12}^{i1}),$$
$$f_{21}^1 = -c_{11}^{1+}(a_{11}^{a1}b_{21}^{1+} + a_{21}^{a1}b_{22}^{1+} - b_{11}^{1+}f_{21}^{i1} - b_{21}^{1+}f_{22}^{i1}),$$
$$f_{22}^1 = -c_{12}^{1+}(a_{11}^{a1}b_{21}^{1+} + a_{21}^{a1}b_{22}^{1+} - b_{11}^{1+}f_{21}^{i1} - b_{21}^{1+}f_{22}^{i1}),$$
$$f_{23}^1 = -c_{13}^{1+}(a_{11}^{a1}b_{21}^{1+} + a_{21}^{a1}b_{22}^{1+} - b_{11}^{1+}f_{21}^{i1} - b_{21}^{1+}f_{22}^{i1}).$$

To calculate matrix $\boldsymbol{B}_0^-$ that is needed for determining the zero level controller, we use the second formula in . As a result, we obtain the expression

$$\boldsymbol{B}_0^- = \boldsymbol{B}_0^+ - \boldsymbol{F}_1\boldsymbol{B}_0^\perp = \begin{pmatrix} b_{11}^m & b_{12}^m & b_{13}^m & 0 \\ b_{21}^m & b_{22}^m & b_{23}^m & 0 \end{pmatrix}.$$

Here

$$b_{11}^m = b_{11}^+ - ((b_{21}b_{32} - b_{22}b_{31})(c_{11}^1f_{11}^1 + c_{21}^1f_{12}^1 + c_{31}^1f_{13}^1))/(b_{11}b_{22} - b_{12}b_{21}),$$
$$b_{12}^m = b_{12}^+ + ((b_{11}b_{32} - b_{12}b_{31})(c_{11}^1f_{11}^1 + c_{21}^1f_{12}^1 + c_{31}^1f_{13}^1))/(b_{11}b_{22} - b_{12}b_{21}),$$
$$b_{13}^m = b_{13}^+ - c_{11}^1f_{11}^1 - c_{21}^1f_{12}^1 - c_{31}^1f_{13}^1,$$
$$b_{21}^m = b_{21}^+ - ((b_{21}b_{32} - b_{22}b_{31})(c_{11}^1f_{21}^1 + c_{21}^1f_{22}^1 + c_{31}^1f_{23}^1))/(b_{11}b_{22} - b_{12}b_{21}),$$
$$b_{22}^m = b_{22}^+ + ((b_{11}b_{32} - b_{12}b_{31})(c_{11}^1f_{21}^1 + c_{21}^1f_{22}^1 + c_{31}^1f_{23}^1))/(b_{11}b_{22} - b_{12}b_{21}),$$
$$b_{23}^m = b_{23}^+ - c_{11}^1f_{21}^1 - c_{21}^1f_{22}^1 - c_{31}^1f_{23}^1.$$

According to Theorem 1, we complete the system of the zero decomposition level using . This yields

$$\boldsymbol{G}_0 = \left(\boldsymbol{B}_0^-\boldsymbol{C}_0^\perp\right)^{+\mathrm{T}}\left(\boldsymbol{B}_0^-\boldsymbol{A}_0\boldsymbol{C}_0^\perp\right)^{\mathrm{T}} = \begin{pmatrix} a_{11}^a & a_{12}^a \\ a_{21}^a & a_{22}^a \end{pmatrix},$$

where

$$a_{11}^a = -b_{11}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m),$$
$$a_{12}^a = -b_{11}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m),$$
$$a_{21}^a = -b_{12}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m),$$
$$a_{22}^a = -b_{12}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m).$$

Now, we should determine Φ_0 for the zero decomposition level. For this purpose, we decompose the matrices $\boldsymbol{H}_0$, $\boldsymbol{G}_0$ of the zero level into two sublevels and calculate the corresponding matrices. We obtain

$$(\boldsymbol{H}_0)_0 = \begin{pmatrix} \dfrac{-b_{21}^m}{b_{11}^m} \\ 1 \end{pmatrix}, \quad (\boldsymbol{H}_0)_0^\perp = \begin{pmatrix} \dfrac{b_{11}^m}{b_{21}^m} & 1 \end{pmatrix},$$

$$(\boldsymbol{H}_0)_0^+ = \begin{pmatrix} \dfrac{-b_{11}^m b_{21}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} \\ \dfrac{(b_{11}^m)^2}{(b_{11}^m)^2 + (b_{21}^m)^2} \end{pmatrix},$$

$$(\boldsymbol{G}_0)_1 = (\boldsymbol{H}_0)_0 \boldsymbol{G}_0 (\boldsymbol{H}_0)_0^{\perp} = a_1^{aa} =$$
$$= -((b_{21}^m)^2 (b_{12}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m) +$$
$$+ (b_{11}^m b_{11}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m))/b_{21}^m))/((b_{21}^m)^2 +$$
$$+ (b_{21}^m)^2) - (b_{11}^m b_{21}^m (b_{12}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m) +$$
$$+ (b_{11}^m b_{11}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m))/b_{21}^m))/((b_{21}^m)^2 + (b_{21}^m)^2),$$

$$(\boldsymbol{H}_0)_1 = (\boldsymbol{H}_0)_0^{\perp} \boldsymbol{G}_0 \boldsymbol{H}_0 = b_1^{bb} =$$
$$= -b_{12}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m) + (b_{21}^m (b_{12}^{0+}(a_{11}b_{11}^m +$$
$$+ a_{21}b_{12}^m + a_{31}b_{13}^m) + (b_{11}^m b_{11}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m +$$
$$+ a_{31}b_{13}^m))/b_{21}^m))/b_{11}^m - (b_{11}^m b_{11}^{0+}(a_{11}b_{21}^m +$$
$$+ a_{21}b_{22}^m + a_{31}b_{23}^m))/b_{21}^m.$$

Using the values of $b_1{}^{bb}$, we then find the matrix (scalar, in this case)

$$(\boldsymbol{H}_0)_1^+ = \frac{1}{b_1^{bb}} = b_1^{bb+}.$$

Let us now assign the eigenvalue as a scalar matrix $(\Phi_0)_1 = \hat{\lambda}_3 = s_{03}$ and calculate the matrix of feedback coefficients for the first sublevel of the zero decomposition level. We obtain

$$k_1 = -b_1^{bb+}(a_1^{aa} - s_{03}).$$

Next, we calculate the matrix

$$(\boldsymbol{H}_0)_0^- = (\boldsymbol{H}_0)_0^+ - k_1 (\boldsymbol{H}_0)_0^{\perp} =$$
$$= \begin{pmatrix} \dfrac{b_1^{bb+} b_{11}^m (a_1^{aa} - s_{03})}{b_{21}^m} - \dfrac{b_{11}^m b_{21}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} & b_1^{bb+}(a_1^{aa} - s_{03}) + \dfrac{b_{11}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} \end{pmatrix}.$$

We specify again the matrix of eigenvalues of the zero sublevel of the zero decomposition level by

$$(\Phi_0)_0 = \hat{\lambda}_4 = s_{02}.$$

Finally, we find matrix $\boldsymbol{K}_0$ by the rule

$$\boldsymbol{K}_0 = (\Phi_0)_0 (\boldsymbol{H}_0)_0^- - (\boldsymbol{H}_0)_0^- \boldsymbol{G}_0 = (K_{11} \quad K_{12}),$$

where in the case under examination

$$K_{11} = -b_{11}^{0+}\left(\frac{b_{11}^m b_{21}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} - \frac{b_1^{bb+} b_{11}^m (a_1^{aa} - s_{03})}{b_{21}^m} \right) \times$$
$$\times (a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m) +$$
$$+ b_{12}^{0+}\left(b_1^{bb+}(a_1^{aa} - s_{03}) + \frac{(b_{11}^m)^2}{(b_{11}^m)^2 + (b_{21}^m)^2} \right) \times$$
$$\times (a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m) -$$
$$- s_{02}\left(\frac{b_{11}^m b_{21}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} - \frac{b_1^{bb+} b_{11}^m (a_1^{aa} - s_{03})}{b_{21}^m} \right),$$

$$K_{12} = s_{02}\left(b_1^{bb+}(a_1^{aa} - s_{03}) + \frac{(b_{11}^m)^2}{(b_{11}^m)^2 + (b_{21}^m)^2} \right) +$$
$$+ b_{12}^{0+}\left(b_1^{bb+}(a_1^{aa} - s_{03}) + \frac{(b_{11}^m)^2}{(b_{11}^m)^2 + (b_{21}^m)^2} \right) \times$$
$$\times (a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m) -$$
$$- b_{11}^{0+}\left(\frac{b_{11}^m b_{21}^m}{(b_{11}^m)^2 + (b_{21}^m)^2} - \frac{b_1^{bb+} b_{11}^m (a_1^{aa} - s_{03})}{b_{21}^m} \right) \times$$
$$\times (a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m).$$

As a result, we obtain using equation , the matrix Φ_0, whose eigenvalues s_{02}, s_{03}, are ensured by the output controller,

$$\Phi_0 = \begin{pmatrix} f_{11}^{i0} & f_{12}^{i0} \\ f_{21}^{i0} & f_{22}^{i0} \end{pmatrix}.$$

Here

$$f_{11}^{i0} = -b_{11}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m) - \frac{b_{21}^m K_{11}}{b_{11}^m},$$
$$f_{12}^{i0} = K_{11} - b_{12}^{0+}(a_{11}b_{11}^m + a_{21}b_{12}^m + a_{31}b_{13}^m),$$
$$f_{21}^{i0} = -b_{11}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m) - \frac{b_{21}^m K_{12}}{b_{11}^m},$$
$$f_{12}^{i0} = K_{12} - b_{12}^{0+}(a_{11}b_{21}^m + a_{21}b_{22}^m + a_{31}b_{23}^m).$$

Further calculations, which were described, for instance, in [9, 10], finally yield the following formula for the output controller vector (28):

$$\boldsymbol{F} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{pmatrix}. \qquad (29)$$

Elements of the matrix can be expressed as

$$f_{11} = b_{12}^m f_{11}^{i0} - a_{22}b_{12}^m - a_{32}b_{13}^m - a_{12}b_{11}^m + b_{22}^m f_{12}^{i0},$$
$$f_{12} = b_{13}^m f_{11}^{i0} - a_{23}b_{12}^m - a_{33}b_{13}^m - a_{13}b_{11}^m + b_{23}^m f_{12}^{i0},$$
$$f_{13} = -a_{14}b_{11}^m,$$
$$f_{21} = b_{12}^m f_{21}^{i0} - a_{22}b_{22}^m - a_{32}b_{23}^m - a_{12}b_{21}^m + b_{22}^m f_{22}^{i0},$$
$$f_{22} = b_{13}^m f_{21}^{i0} - a_{23}b_{22}^m - a_{33}b_{23}^m - a_{13}b_{21}^m + b_{23}^m f_{22}^{i0},$$
$$f_{13} = -a_{14}b_{21}^m.$$

The synthesized controller (and the control system based on it) ensures exactly the specified spectrum (8) for controlled lateral motion of the SH. This assertion can be directly checked with the help of appropriate analytical calculations. For this purpose it is sufficient to make use of the package Symbolic Toolbox MATLAB; namely, one can use the *eig* instruction to calculate the eigenvalues of the $\boldsymbol{A} + \boldsymbol{BFC}$ matrix.

5. Numerical analysis

Let use for simulation of the lateral motion of the hypothetical SH the following numerical values of the coefficient matrices:

$$\boldsymbol{A} = \begin{pmatrix} -0,1900 & -6,2000 & 68,9161 & -9,7932 \\ -0,1200 & -6,2519 & -0,1900 & 0 \\ -0,0500 & 0,1000 & -0,8720 & 0 \\ 0 & 1 & 0,1000 & 0 \end{pmatrix}, \quad (30)$$

$$\boldsymbol{B} = \begin{pmatrix} -16,1744 & -6,0409 \\ -135,4887 & -2,3329 \\ 3,5087 & -13,0006 \\ 0 & 0 \end{pmatrix}. \quad (31)$$

Suppose that we want the closed-loop system "SH + + control system" with matrices (30), (31) to have the following specified spectrum (8):

$$\Lambda = \{-1,5;\ -1,5;\ -1,5;\ -1,5\}. \quad (32)$$

The set (32), as we can see, consists of identical numbers, i. e., we want the closed-loop system "SH + + control system" to have the spectrum with a multiplication factor of 4.

It should be noted that, even under much simpler conditions of the closed-loop control synthesis, when all the elements of the state vector are accessible for measurement, known methods do not allow this problem to be solved.

For instance, a well-known function *place* from the MATLAB software package will deliver an error in this case, since it is required to ensure that the multiplicity of the spectrum elements are greater than the number of inputs.

For the numerical values of the matrices (30), (31), and the desired spectrum (32) with use of equation (29), we obtain the controller matrix:

$$\boldsymbol{F} = \begin{pmatrix} -0,0299 & -0,0272 & 0,0168 \\ -0,0125 & 0,2060 & -0,0128 \end{pmatrix}. \quad (33)$$

The matrix $\boldsymbol{A} + \boldsymbol{BFC}$ of the close-loop system "SH + + control system" will take the form of

$$\boldsymbol{A} + \boldsymbol{BFC} = \begin{pmatrix} -0,1990 & -5,6402 & 68,1118 & -9,9879 \\ -0,1200 & -2,1649 & 3,0143 & -2,2500 \\ -0,0500 & 0,1572 & -3,6451 & 0,2267 \\ 0 & 1 & -0,4663 & 0 \end{pmatrix}. \quad (34)$$

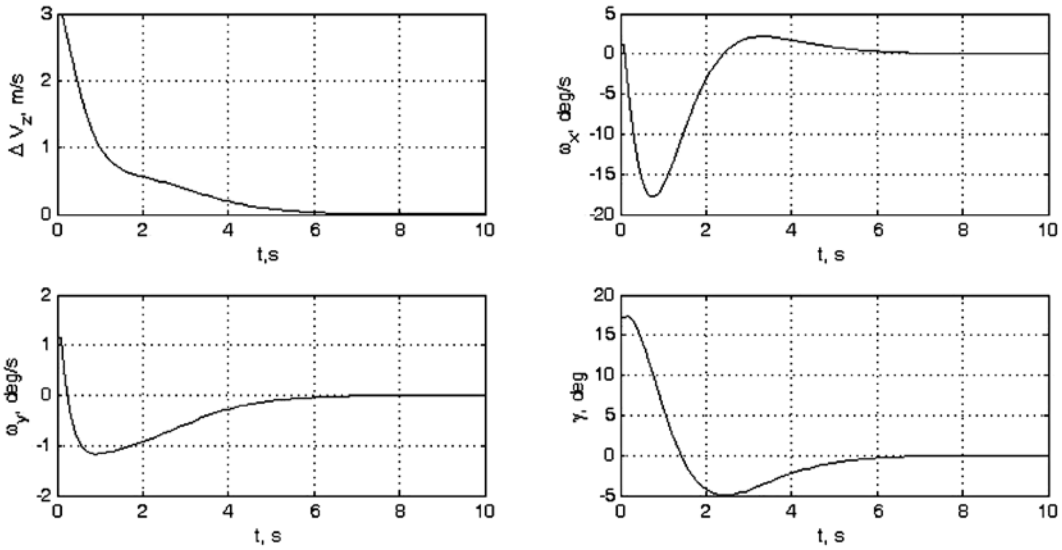

Fig. 1. The diagrams of transition functions for the state vector components of the close-loop "SH + control system"

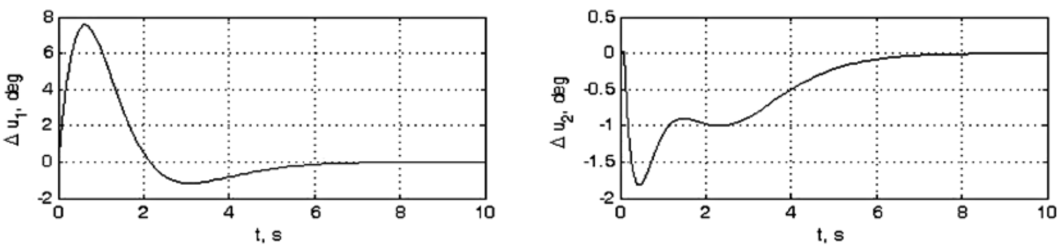

Fig. 2. The values of control actions of the close-loop "SH + control system"

The computation of eigenvalues of the matrix $\boldsymbol{A} + \boldsymbol{BFC}$ yields

$$\operatorname{eig}\left(\boldsymbol{A}+\boldsymbol{BFC}\right)=\{-1{,}5;\ \ -1{,}5;\ \ -1{,}5;\ \ -1{,}5\},$$

which coincides with the set (32), which we wanted to obtain.

For the initial values of the SH state vector in the system of SI units that are

$$\begin{aligned}&\left(\Delta V_z \quad \Delta\omega_x \quad \Delta\omega_y \quad \Delta\gamma\right)^{\mathrm{T}} = \\ &=\left(3{,}00 \quad 0{,}02 \quad 0{,}02 \quad 0{,}30\right)^{\mathrm{T}},\end{aligned}$$

the diagrams of transition functions for the state vector components of the close-loop "SH + control system" are provided in Fig. 1. Correspondingly, the values of control actions are shown in Fig. 2. We can see that the transition processes are fast-decaying ones and have a close-to-aperiodic (low-oscillatory) type, which ensures good handling qualities of the vehicle.

6. Conclusions

The problem of a stabilization law synthesis of single-airscrew helicopter's lateral motion for lack of information about the lateral speed of its motion has been analytically solved. The solution is based on the method of the output signal control synthesis that provides a specified spectrum of the MIMO-system's motion, presented earlier in [11]. The method is based on a decomposition of the system using orthogonal transformations. The method has no restrictions on the algebraic and geometric multiplicities of the spectrum elements, and also makes it possible to obtain analytical solutions and a parameterization (construction) of a set of controllers. Numerical simulation data that confirm the analytical expressions are also presented.

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