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The New Analytical Algorithm for Determining the Strapdown INS Orientation

Abstract

The analytical solution of an approximate (truncated) equation for the vector of a rigid body finite rotation has made it possible to solve the problem of determining the quaternion of orientation of a rigid body for an arbitrary angular velocity and small angle of rotation of a rigid body with the help of quadratures. Proceeding from this solution, the following approach to the construction of the new analytical algorithm for computation of a rigid body orientation with the use of strapdown INS is proposed: 1) By the set components of the angular velocity of a rigid body on the basis of mutually — unambiguous changes of the variables at each time point, a new angular velocity of a rigid body is calculated; 2) Using the new angular velocity and the initial position of a rigid body, with the help of the quadratures we find the exact solution of an approximate linear equation for the vector of a rigid body finite rotation with a zero initial condition; 3) The value of the quaternion orientation of a rigid body (strapdown INS) is determined by the vector of finite rotation. During construction of the algorithm for strapdown INS orientation at each subsequent step the change of the variables takes into account the previous step of the algorithm in such a way that each time the initial value of the vector of finite rotation of a rigid body will be equal to zero. Since the proposed algorithm for the analytical solution of the approximate linear equation for the vector of finite rotation is exact, it has a regular character for all angular motions of a rigid body).

Keywords: *analytical solution, algorithm, orientation, vector of finite rotation, arbitrary angular velocity, rigid body, strapdown INS, quaternion*

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Новый аналитический алгоритм определения ориентации БИНС[1]

На основе полученного точного решения приближенного (усеченного) уравнения для вектора конечного поворота твердого тела с помощью квадратур решена задача определения кватерниона ориентации твердого тела при произвольном векторе угловой скорости и малом угле поворота твердого тела. Исходя из этого решения предложен

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следующий подход к построению нового аналитического алгоритма для вычисления ориентации твердого тела с помощью БИНС: 1) по заданным компонентам вектора угловой скорости твердого тела на основе взаимно-однозначных замен переменных в каждый момент времени вычисляется новый вектор угловой скорости некоторой новой введенной системы координат; 2) используя новый вектор угловой скорости и начальное положение твердого тела, с помощью квадратур находится точное решение приближенного линейного уравнения для вектора конечного поворота с нулевым начальным условием; 3) по вектору конечного поворота определяется значение кватерниона ориентации твердого тела (БИНС). Отметим, что при построении алгоритма ориентации БИНС на каждом последующем шаге замена переменных учитывает предыдущий шаг алгоритма таким образом, что начальное значение вектора конечного поворота твердого тела каждый раз будет нулевым. Поскольку предлагаемый алгоритм аналитического решения приближенного линейного уравнения для вектора конечного поворота твердого тела является точным, он носит регулярный характер при всех угловых движениях твердого тела.

Ключевые слова: *аналитическое решение, алгоритм, ориентация, вектор конечного поворота, произвольная угловая скорость, твердое тело, БИНС, кватернион*

Introduction

During operation of many strapdown inertial navigation systems (SINS) the vector of a rigid body finite rotation is periodically calculated by the method of approximate solution of the approximate linear differential equation for the vector of finite rotation (in the theory and practice of SINS construction, in ultra rapid cycles of algorithms for small angles of rotation, the nonlinear term in the differential equation for the vector of finite rotation of a rigid body is neglected). The angular velocity vector of a rigid body is the input quantity in the equation. Note that the full nonlinear differential equation for the vector of finite rotation of a rigid body is an analog of the quaternion linear equation; the vector and the quaternion of the rigid body orientation are linked by known relations. The approximate linear differential equation for the vector of finite rotation in the literature is solved by various numerical methods, for example, by Picard's method, then the second iteration of this method in the practice of SINS can be taken for the final one. This term in the iteration formula of Picard's method is called a non-commutative rotation vector, or "coning". For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The study of non-commutative rotations (or "coning") as a kind of mechanical motion of bodies, separation of numerical algorithms for determining the orientation of a rigid body (SINS) for rapid and slow counting cycles are aimed at compensation for the effect of this phenomenon. Meanwhile, for some new angular velocity vector, which is obtained in determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector in unambiguous interchanges of variables in the motion equations for a rigid body, the approximate differential equation for the vector of finite rotation of a rigid body admits of an exact analytic solution. We will show this.

The problem is to define the quaternion of orientation $\mathbf{\Lambda}$ of a rigid body with respect to an arbitrary given angular velocity vector $\boldsymbol{\omega}(t)$ and the initial angular position of a rigid body in space based on the quaternion kinematic equation known as the Darboux problem. Further, we make changes of variables by the scheme $\mathbf{\Lambda} \to \mathbf{U}$, where $\mathbf{U}$ is the quaternion of the orientation of some introduced coordinate system, it is always possible to reverse the transition $\mathbf{U} \to \mathbf{\Lambda}$. These changes have the character of rotation transformations and reduce the initial problem of determining the orientation of a rigid body (quaternion $\mathbf{\Lambda}$) with an arbitrary variable angular velocity vector $\boldsymbol{\omega}(t)$ to the problem where the angular velocity vector $\mathbf{w}(t)$ of the introduced coordinate system, remaining generally variable in absolute value, performs a definite motion — rotates around one of the axes of the coordinate system. This motion is generalized conical precession and agrees well with the known Poinsot's concept that any rigid body rotation about a fixed point can be represented as a conical motion. Finding an analytical solution of the quaternion differential equation obtained with respect to the new unknown quaternion $\mathbf{U}$ is still a difficult problem. However, the equation differing from this only by the coefficient "1/2" in the right-hand side (i.e., with the angular velocity vector $\mathbf{\Omega}(t)/2$) is solved in closed form. Moreover, we note that the quaternion differential equation is isomorphic to the homogeneous vector differential equation of Poisson.

The resulting problem with the angular velocity vector $\mathbf{\Omega}(t)$ and the unknown quaternion of orientation $\mathbf{U}$ is associated with the complete nonlinear differential equation with respect to the unknown vector of finite rotation of a rigid body $\mathbf{x}$. The approximate linear equation for the vector of finite rotation, which is an inhomogeneous vector differential equation whose homogeneous part is equivalent to the Poisson equation with the vector coefficient $\mathbf{\Omega}(t)/2$, becomes analytically solvable and its solution $\mathbf{x}^*$ is obtained in quadratures by the Lagrange method.

The exact solution of the approximate linear equation for the vector of finite rotation of a rigid body made it possible to solve the problem of determining the quaternion of orientation of a rigid body for an arbitrary angular velocity and small angle of rotation of a rigid body with the help of quadratures. Proceeding from this solution, the following approach to the design of a new algorithm for computation of SINS orientation is proposed: 1) by the set components of the angular velocity of a rigid body on the basis of unambiguous interchanges of the variables at each time point, a new angular velocity $\boldsymbol{\Omega}(t)$ of some new coordinate system is calculated; 2) using the new angular velocity and the initial position of a rigid body, we find the exact solution $\mathbf{x}^*$ of the approximate linear equation for the vector of finite rotation with a zero initial condition with the help of quadratures; 3) the value of the quaternion orientation of a rigid body (SINS) is determined by the vector of finite rotation on the scheme $\mathbf{x}^* \approx \mathbf{x} \Leftrightarrow \mathbf{U} \to \boldsymbol{\Lambda}$.

During construction of the algorithm for SINS orientation at each subsequent step the change of the variables takes into account the previous step of the algorithm in such a way that each time the initial value of the vector of finite rotation of a rigid body will be equal to zero. Since the proposed algorithm for the analytical solution of the approximate linear equation for the vector of finite rotation is exact, it has a regular character for all angular motions of a rigid body.

Previously, the authors constructed the exact solution of the Bortz approximate equation for the orientation vector of a rigid body and the quaternion orientation algorithm of SINS on its basis [1, 2].

1. Statement of the problem of determining the orientation of a rigid body (SINS)

Consider the Cauchy problem for quaternion kinematic equation [3] with arbitrary given angular velocity vector-function $\boldsymbol{\omega}(t)$, written in the following form (this problem is known as the Darboux problem):

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega}(t), \quad (1.1)$$

$$\boldsymbol{\Lambda}(t_0) = \boldsymbol{\Lambda}_0. \quad (1.2)$$

Here $\boldsymbol{\Lambda}(t) = \lambda_0(t) + \lambda_1(t)i_1 + \lambda_2(t)i_2 + \lambda_3(t)i_3$ is a quaternion describing the position of a rigid body in an inertial space; $\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{i}_1 + \omega_2(t)\mathbf{i}_2 + \omega_3(t)\mathbf{i}_3$ is the angular velocity vector of the rigid body specified by its projections onto body-fixed coordinate axes; i_1, i_2, i_3 — the units of the hypercomplex space (imaginary Hamiltonian units), which can be identified with the vectors of a three-dimensional vector space $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$; the symbol "$\circ$" stands for the quaternion product; $\boldsymbol{\Lambda}_0$ is the initial value of the quaternion $\boldsymbol{\Lambda}(t)$ at $t = t_0$, $t \in [t_0, \infty)$ (t_0 set equal to 0). The problem is to find the quaternion $\boldsymbol{\Lambda}(t)$.

The problem of determining of the vector of a rigid body finite rotation $\mathbf{x}(t)$ [3] relative to an inertial space can also be posed by solving the exact differential equation for the vector of finite rotation of a rigid body

$$\dot{\mathbf{x}} = \boldsymbol{\omega} + \mathbf{x} \times \boldsymbol{\omega}/2 + (\mathbf{x}, \boldsymbol{\omega})\mathbf{x}/4, \quad (1.3)$$

where "$\times$" and "$(\cdot,\ \cdot)$" mean the vector and the scalar products. In equation (1.3) the input quantity is the angular velocity vector $\boldsymbol{\omega}$. Note that the nonlinear equation (1.3) for the vector of finite rotation of a rigid body $\mathbf{x}$ is an analogue of the quaternion linear equation (1.1); vector $\mathbf{x}$ and quaternion $\boldsymbol{\Lambda}$ are connected by the relations:

$$\begin{aligned} &\mathbf{x} = 2\boldsymbol{\lambda}_v/\lambda_0 = 2\mathbf{e}\operatorname{tg}(\varphi/2),\ \mathbf{e} = \boldsymbol{\lambda}_v/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \\ &\boldsymbol{\lambda}_v = \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, \\ &\cos\varphi = \lambda_0,\ \sin\varphi = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2},\ 0 \leqslant \varphi < \pi, \end{aligned} \quad (1.4)$$

where φ is the angle of orientation of a rigid body and $\mathbf{e}$ — the Euler axis of rotation. In the practice of constructing of SINS orientation algorithms by numerical solution of equation (1.3) on a time interval $t_{m-1} \leqslant t < t_m$ the third member in this equation is neglected for small angles of rotation (it is the magnitude of the second order). If the derived simplified (approximate) differential equation

$$\dot{\mathbf{x}}^* = \boldsymbol{\omega} + \mathbf{x}^* \times \boldsymbol{\omega}/2 \quad (1.5)$$

is solved by Picard's iterative method, then the second iteration of this method is taken for the final one [4, 5]:

$$\mathbf{x}_m^* = \int_{t_{m-1}}^{t_m} (\boldsymbol{\omega}(t)dt + \boldsymbol{\alpha}(t) \times \boldsymbol{\omega}(t)/2)dt = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m,$$

$$\boldsymbol{\alpha}(t) = \int_{t_{m-1}}^{t_m} \boldsymbol{\omega}(\tau)d\tau,\ \boldsymbol{\alpha}_m = \boldsymbol{\alpha}(t_m),$$

$$\boldsymbol{\beta}(t) = \int_{t_{m-1}}^{t_m} \boldsymbol{\alpha}(\tau) \times \boldsymbol{\omega}(\tau)d\tau/2,\ \boldsymbol{\beta}_m = b(t_m),$$

where vector $\boldsymbol{\beta}$ is called a non-commutative rotation vector, or "coning". For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The study of non-commutative rotations (or "coning") as a kind of mechanical motion of bodies, separation of numerical algorithms for determining the orientation of a rigid body SINS) for rapid and slow counting cycles are aimed at compensation for the effect of this phenomenon. Meanwhile, for some new angular velocity vector $\mathbf{w}(t)$, which is obtained in determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector $\boldsymbol{\omega}(t)$ in unambiguous replacements of variables in the motion equations for a rigid body, the approximate equation for the vector of finite rotation of a rigid body admits of an exact analytic solution, which will be shown in what follows.

2. The exact solution of the approximate equation for the vector of finite rotation of a rigid body and design of the algorithm for determining SINS orientation on its basis

Let's write unambiguous replacements of variables in the problem (1.1), (1.2) [6] according to the scheme $\boldsymbol{\Lambda} \to \mathbf{U}$, where $\mathbf{U}(t)$ is the quaternion of orientation of some introduced coordinate system (new variable), quaternion $\mathbf{V}(t)$ is the generated transition operator, $\mathbf{K}$ is an arbitrary constant quaternion:

$$\boldsymbol{\Lambda}(t) = \mathbf{U}(t) \circ \mathbf{K} \circ \mathbf{V}(t), \ \|\mathbf{K}\| = \|\mathbf{V}\| = 1, \quad (2.1)$$

$$\mathbf{V}(t) = \left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) \circ \circ \exp\left(\mathbf{i}_3 \mathrm{N}(t)/2\right) \circ \exp\left(\mathbf{i}_1 \Omega_1(t)/2\right), \quad (2.2)$$

$$2\dot{\mathbf{U}} = \mathbf{U} \circ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}, \quad (2.3)$$

$$\mathbf{w}(t) = \mu(t)\left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) - 2\mathbf{i}_3 \nu(t), \quad (2.4)$$

$$\mu(t) = \omega_2(t) \cos \Omega_1(t) - \omega_3(t) \sin \Omega_1(t),$$
$$\nu(t) = \omega_2(t) \sin \Omega_1(t) + \omega_3(t) \cos \Omega_1(t), \quad (2.5)$$
$$\mathrm{N}(t) = \int_0^t \nu(\tau) d\tau, \quad \Omega_1(t) = \int_0^t \omega_1(\tau) d\tau,$$

$$\mathbf{U}(0) = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \tilde{\mathbf{K}} \quad (2.6)$$

where (2.3)—(2.6) the new problem for determining the orientation of a rigid body with the new angular velocity vector $\mathbf{w}(t)$, "$\|\cdot\|$" means quaternion norm.

Finding an analytical solution to the resulting quaternion differential equation (2.3) remains a difficult task. However, the equation that differs from this one only in the coefficient "1/2" on the right side (i.e. with the angular velocity vector $\mathbf{w}(t)/2$)

$$2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}/2, \quad (2.7)$$

$$\boldsymbol{\Psi}(0) = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \tilde{\mathbf{K}} \quad (2.8)$$

is solved in a closed form. Choose quaternion $\mathbf{K}$ in the form $\mathbf{K} = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2)$ so that the initial conditions (2.6), (2.8) become unit $\mathbf{U}(0) = \boldsymbol{\Psi}(0) = 1$. Note that this technique with quaternion $\mathbf{K}$ is important in the subsequent construction of the algorithm of SINS orientation. The solution of the Cauchy problem (2.7), (2.8) will be written as follows:

$$\boldsymbol{\Psi} = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \boldsymbol{\Phi}(t) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0, \quad (2.9)$$

$$\boldsymbol{\Phi}(t) = \exp\left(\mathbf{i}_2 \mathrm{M}(t)/4\right) \circ \exp\left(-\mathbf{i}_3 \mathrm{N}(t)/2\right),$$
$$\mathrm{M}(t) = \int_0^t \mu(\tau) d\tau. \quad (2.10)$$

We check the correctness of the obtained solution of the problem (2.7), (2.8) by differentiating the expression (2.9) taking into account (2.10)

$$\dot{\boldsymbol{\Psi}}(t) = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \boldsymbol{\Phi}(t) \circ (\mu(t) \exp\left(\mathbf{i}_3 \mathrm{N}(t)/2\right) \circ$$
$$\circ\, \mathbf{i}_2 \circ \exp\left(-\mathbf{i}_3 \mathrm{N}(t)/2\right)/4 - \mathbf{i}_3 \nu(t)/2) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0 =$$
$$= \boldsymbol{\Psi}(t) \circ \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ (\mu(t)\left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) -$$
$$- 2\mathbf{i}_3 \nu(t)) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0/4$$

or otherwise

$$2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{w}(t) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0/2$$

herewith $\boldsymbol{\Psi}(0) = 1$, that matches with expressions (2.7), (2.8).

On the basis of expressions of type (2.4) we associate the reduced quaternion problem of determining orientation (2.3)—(2.6) with the problem with the vector approximate differential equation of the type (1.5):

$$\dot{\mathbf{x}}^* = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{w}(t) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0 + + \mathbf{x}^* \times (\boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{w}(t) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0)/2, \quad (2.11)$$

$$\mathbf{x}^*(0) = 0. \quad (2.12)$$

We note that the homogeneous part of the vector linear differential equation (2.11) is equivalent to the solvable system (2.7) written in the form of a vector differential Poisson equation. From the Lagrange method of solving linear inhomogeneous differential systems of equations, the exact solution of the approximate equation (2.11) will have the form on the basis of (2.9), (2.10)

$$\mathbf{x}^* = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \tilde{\boldsymbol{\Phi}}(t) \circ \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t) \circ \mathbf{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0. \quad (2.13)$$

We check the correctness of the obtained solution of the equation (2.12), (2.8) by differentiating the expression (2.13):

$$\dot{\mathbf{x}}^* = \mathbf{K} \circ (\tilde{\mathbf{\Phi}}(t) \circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ \mathbf{\Phi}(t) \circ \mathbf{w}(t) -$$

$$- \mathbf{w}(t) \circ \tilde{\mathbf{\Phi}}(t) \circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ \mathbf{\Phi}(t)) \circ \tilde{\mathbf{K}}/4 +$$

$$+ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}} = \mathbf{K} \circ (\tilde{\mathbf{\Phi}}(t) \circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ$$

$$\circ \mathbf{\Phi}(t) \circ \tilde{\mathbf{K}} \circ \mathbf{K} \circ \mathbf{w}(t) - \mathbf{w}(t) \circ \tilde{\mathbf{K}} \circ \mathbf{K} \circ \tilde{\mathbf{\Phi}}(t) \circ$$

$$\circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ \mathbf{\Phi}(t)) \circ \tilde{\mathbf{K}}/4 +$$

$$+ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}} = \mathbf{x}^* \times (\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}})/2 + \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}.$$

Thus, the problem of determining the orientation of a rigid body (1.1)—(1.3) on the basis of (1.5) at small rotation angles is completely solved with the help of quadratures. We give the analytical algorithm for determining orientation of a rigid body (SINS) at arbitrary angles of rotation:

1) using the components of angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body, functions $\mu(t)$, $\nu(t)$ are calculated at each moment of time t by the formulas:

$$\Omega_1(t) = \int_0^t \omega_1(\tau) d\tau,$$
$$\mu(t) = \omega_2(t) \cos \Omega_1(t) - \omega_3(t) \sin \Omega_1(t), \quad (2.14)$$
$$\nu(t) = \omega_2(t) \sin \Omega_1(t) + \omega_3(t) \cos \Omega_1(t);$$

2) vector $\mathbf{w}(t)$ is determined by the calculated $\mu(t)$, $\nu(t)$:

$$\mathrm{N}(t) = \int_0^t \nu(\tau) d\tau,$$
$$\mathbf{w}(t) = \mu(t)\left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) - 2\mathbf{i}_3 \nu(t); \quad (2.15)$$

3) the approximate value of the vector of finite rotation of a rigid body $\mathbf{x}^*$ is calculated using vector $\mathbf{w}(t)$ and the initial position of rigid body $\mathbf{\Lambda}_0$:

$$\mathrm{M}(t) = \int_0^t \mu(\tau) d\tau,$$
$$\mathbf{\Phi}(t) = \exp\left(\mathbf{i}_2 \mathrm{M}(t)/4\right) \circ \exp\left(-\mathbf{i}_3 \mathrm{N}(t)/2\right), \quad (2.16)$$

$$\mathbf{x}^* = \mathbf{K} \circ \tilde{\mathbf{\Phi}}(t) \circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ \mathbf{\Phi}(t) \circ \tilde{\mathbf{K}},$$
$$\mathbf{K} = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2); \quad (2.17)$$

4) the components of quaternion $\mathbf{U}$ are determined by the vector $\mathbf{x}^*$ on the basis of formulas of the type (1.4);

5) an approximate value of quaternion of a rigid body (SINS) orientation $\mathbf{\Lambda}^{approx}$ is obtained

$$\mathbf{\Lambda}^{approx} = \mathbf{U}(t) \circ \mathbf{K} \circ \left(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)\right) \circ$$
$$\circ \exp\left(\mathbf{i}_3 \mathrm{N}(t)/2\right) \circ \exp\left(\mathbf{i}_1 \Omega_1(t)/2\right).$$

Quaternion $\mathbf{K}$ should be selected in the form $\mathbf{K}_m = \mathbf{\Lambda}_{m-1} \circ (-\mathbf{i}_2)$ when implementing the SINS orientation algorithm at each subsequent step m of algorithm. Then the initial value of variable $\mathbf{x}^*$ will be zero each time.

Conclusion

In contrast to the algorithms for determining the orientation of an object described in [4, 5, 7] using approximate numerical solutions of a truncated equation for a vector of orientation of a rigid body and reading information about the angular velocity of an object directly from sensing elements of SINS, the essence of the approach proposed in the article is that by first transforming this information using formulas (2.14), (2.15), the equation for the vector of the final rotation of a rigid body becomes clearly solvable by formulas (2.17). The quaternion on the basis of which the solution of the problem is built is written in elementary functions and quadratures by formulas (2.16).

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