УДК 681.511.46 + 681.5.013 DOI: 10.17587/mau.19.755-761

A. R. Gaiduk, D. Sc., Professor, gaiduk_2003@mail.ru,
Southern Federal University, Department of Control Systems, Taganrog, 347920, Russian Federation

Corresponding author: **Gaiduk Anatoly R.,** D. Sc., Professor,
Southern Federal University, Taganrog, 347920, Russian Federation,
e-mail: gaiduk_2003@mail.ru

Accepted on August 22, 2018

Nonlinear Control Systems Design by Transformation Method

Abstract

The analytical approaches to design of nonlinear control systems by the transformation of the nonlinear plant equations into quasilinear forms or into Jordan controlled form are considered. Shortly definitions of these forms and the mathematical expressions necessary for design of the control systems by these methods are submitted. These approaches can be applied if the plant's nonlinearities are differentiable, the plant is controllable and the additional conditions are satisfied. Procedure of a control system design, i.e. definition of the equations of the control device, in both cases is completely analytical. Desirable quality of transients is provided with that, that corresponding values are given to roots of the characteristic equations of some matrixes by calculation of the nonlinear control. The proposed methods provide asymptotical stability of the equilibrium in a bounded domain of the state space or its global stability and also desirable performance of transients. Performance of the nonlinear plants equations in the quasilinear form has no any complexities, if the mentioned above conditions are satisfied. The transformation of these equations to the Jordan controlled form very much often is reduced to change of the state variables designations of the plants. The suggested methods can be applied to design of control systems by various nonlinear technical plants ship-building, machine-building, aviation, agricultural and many other manufactures. Examples of the control systems design by the proposed analytical methods are given.

Keywords: *nonlinear plant, transformation, quasilinear form, Jordan controlled form, controllability, design, control system*

Acknowledgements: The article is prepared on the basis of results of performance in SFedU of the project No. 16-08-00013-a under the RFBR grant.

For citation:

Gaiduk A. R. Nonlinear Control Systems Design by Transformation Method, *Mekhatronica, Avtomatizatsiya, Upravlenie*, 2018, vol. 19, no. 12, pp. 755—761.

DOI: 10.17587/mau.19.755-761

Introduction

The problem of control systems design for nonlinear plants still has no exhaustive solution though it was considered in many works. However, majority proposed methods have the bounded scope. Therefore the new control design methods for nonlinear plants are actually. The transformation method of the plant equations to some simple form is widely used for solution of the linear and nonlinear control systems problem. This approach simplifies the solution of this problem and makes it analytical. In nonlinear cases the plant equations are transformed to the various forms. It can be normal canonical control form [1—4], triangular form [5, 6], Lukyanov—Utkin regular form [7], quasilinear form [8—10], Jordan controlled form (JCF) [11—13] and others. Transformation of the humanoid robot equations to a controllability canonical form has allowed designing a control system which compensates influence of external disturbances [2, 3] and parameters uncertainties [3]. If the plant equations are represented in the triangular form it is easy to apply the backstepping method to design stabilizing control or to design an adaptive control [5, 6]. The Lukyanov—Utkin regular form of the plant equations allows decoupling the high dimension design problem on several tasks of the smaller dimensions [7].

The purpose of this paper is the representation of a rather effective analytically approaches to the design of nonlinear control systems on the basis of transformation of the plants and control systems equations to the quasilinear or Jordan controlled form. These approaches allow analytically finding controls, which provides stability of the system equilibrium [9], duration and character of transients and astatic or full compensation of the influence of bounded external disturbances [12, 13]. Mathematical basis of these approaches are differentiability plant nonlinearities, plant controllability [14]. Application of the considered methods is expedient as the equations of many real plants have differentiable nonlinearity, and plant equations have JCF or may be represented in this form by simple transformation.

This article is organized as follows. The problem of nonlinear control systems design is given in section 2. Transformation of the nonlinear equations of dynamic plants and systems to the quasilinear form is presented in section 3. Features of this transformation and the suggested analytical method of nonlinear control systems design on the basis of this form are given in section 4. The approach to design problem of the nonlinear control systems on the basis of a Jordan controlled form is considered in section 5. In the final section the corresponding examples of control systems design are resulted.

Statement of the Control Systems Design Problem

Let the equation of some controlled plant looks like

$$\dot{x} = f(x, u_0), \tag{1}$$

where x is a measured state n-vector; $f(x, u)$ is a nonlinear differentiable n-vector-function; $u_0 = u_0(x)$ is an scalar control. Assume $x^\circ = x^\circ(t, u_0^\circ)$ is the vector that describes the unperturbed motion of the plant (1); u_0° is the corresponding control. Enter the deviations $\tilde{x} = x - x^\circ$ and $u = u_0 - u_0^\circ$. Then the plant (1) in deviations is described by the equation:

$$\dot{\tilde{x}} = f(\tilde{x}, u), \tag{2}$$

where $f(\tilde{x}, u) = [f(x^\circ + \tilde{x}, u_0^\circ + u) - f(x^\circ, u_0^\circ)]$ is a nonlinear differentiable vector-function.

Usually, when $u = 0$ the equilibrium $\tilde{x} \equiv 0$ of the plant (2) is unstable or the processes in this plant are unsatisfactory. The control system design problem consists in the definition of the control $u = u(\tilde{x})$ so that equilibrium $\tilde{x} \equiv 0$ of the plant (2) was asymptotically stable, at least, in an bounded domain Ω, i.e.

$$\lim_{t\to\infty} \tilde{x}(t, \tilde{x}_0, u(\tilde{x})) = 0, \quad \tilde{x}_0 \in \Omega_0 \in \Omega \in R^n,\ \tilde{x} \in \Omega \in R^n, \tag{3}$$

where Ω_0 is a bounded attraction domain of the equilibrium $\tilde{x} \equiv 0$. This control should provide also the desired duration and character of transients.

The solution method of the considered design problem depends, first of all, on properties of the nonlinear vector-function $f(\tilde{x}, u)$. In the beginning we shall consider as the quasilinear form of the equation (2) is applied to this purpose and then — the Jordan controlled form. Conditions on the nonlinear vector-function $f(\tilde{x}, u)$, at which the design problem has the solution by there methods, will be shown.

Control System Design Using Quasilinear Form

Suppose the nonlinear vector-function $f(\tilde{x}, u)$ in equation (2) such that

$$f(0, 0) = 0;\ \frac{\partial f_i(\tilde{x}, u)}{\partial u} = f_{iu}(\tilde{x}); \quad \tilde{x} \in \Omega \in R^n,\ \|\tilde{x}\| \leqslant M_\Omega < \infty. \tag{4}$$

where M_Ω is a number dependent on the sizes of domain Ω. Before passing to the solution of the statement problem, we shall define the term "quasilinear form" of nonlinear functions and nonlinear vector-functions.

Assume some nonlinear function $f(x) = f(x_1, ..., x_n)$ of variables $x_1, ..., x_n$ is differentiable. Then it can be presented always as follows:

$$f(x) = a^T(x)x + f(0) = [a_1(x)\ ...\ a_n(x)]x + f(0), \tag{5}$$

where $a^T(x)$ is some functional n-vector and $a_i(x)$ are its components depend on a way of integration of the partial derivatives $f_i(x) = \partial f(x)/\partial x_i$ from a point $x \equiv 0$ to a point x. Various ways of integration give various quasilinear representations of the nonlinear function [8, 9, 14]. We shall use the following expressions for definition of the components $a_i(x)$:

$$a_i^I(x) = \int_0^1 f_i(x_1, \ldots x_{i-1}, \theta x_i, 0, \ldots 0) d\theta,\ i = \overline{1,\ n}. \tag{6}$$

The validity of the expressions (5), (6) will be shown on examples. Let us consider the function $f_*(x) = x_1 x_2^2 + x_2 x_3^3 + x_1^4 x_3 + \upsilon$, where x_1, x_2, x_3 there are independent variables and υ is some constant. The function $f_*(x)$ is differentiable; therefore there are its partial derivatives:

$$f_{*1}(x) = x_2^2 + 4x_1^3 x_3; \quad f_{*2}(x) = 2x_1 x_2 + x_3^3;\ f_{*3}(x) = 3x_2 x_3^2 + x_1^4, \tag{7}$$

and $f_*(0) = \upsilon$. Substituting the received the partial derivatives (7) in the formula (6), we shall find: $a_{*1}(x) = 0$, $a_{*2}(x) = x_1x_2$, $a_{*3}(x) = x_1^4 + x_2x_3^2$, i.e. the vector $a_*^T(x) = [0 \quad x_1x_2 \quad x_1^4 + x_2x_3^2]$. The received vector and the formula (5) give the consider nonlinear function $f_*(x)$.

The expressions (5) and (6) are fair and in relation to differentiable vector-functions with replacement of a vector $a^T(x)$ by a corresponding functional matrix. Let, for example, $x^T = [x_1\ x_2\ x_3]$ and $f_{**}^T(x) = [3x_2 + 4x_1^2 \quad 7x_2x_3 + 2\sin x_1 \quad 1,2x_3x_1 + x_3^3]$. The expression (6) applied to the components of this vector-function give a matrix

$$A(x) = \begin{bmatrix} 4x_1 & 3 & 0 \\ 2\omega(x_1) & 0 & 7x_2 \\ 0 & 0 & 1,2x_1 + x_3^2 \end{bmatrix},$$

where $\omega(x_1) = (\sin x_1)/x_1$. The validity of the expression $f_{**}(x) = A(x)x + f_{**}(0)$ is evidently.

Though quasilinear representations of a differentiable function and an vector-function are not unique, but any quasilinear form describes the given nonlinearity *precisely*, in difference, for example, from "the first approximation" [4, 5].

The vector-function $f(\tilde{x}, u)$ from the equation (2) satisfies the conditions (4), therefore according to expressions (5), (6) this equation can be submitted as follows:

$$\dot{\tilde{x}} = A(\tilde{x})\tilde{x} + b(x)u,\ \tilde{x} \in \Omega \in R^n,\ \|\tilde{x}\| \leqslant M_\Omega < \infty. \quad (8)$$

Here $A(\tilde{x}) = [a_{ij}(\tilde{x})]$ is a functional $n\times n$-matrix, $b(\tilde{x}) = [b_i(\tilde{x})]$ is functional n-vector and $b_i(\tilde{x}) = f_{iu}(\tilde{x})$. The equation (8) is the quasilinear form of the equation (2).

In summary, we shall emphasize: the right parts of the equations (2) and (8) are completely identical with all $\tilde{x} \in \Omega \in R^n, \|\tilde{x}\| \leqslant M_\Omega < \infty$, i.e. the quasilinear form (8) is the exact representation of the nonlinear differential equations such as (2), satisfying conditions (4). The quasilinear form of nonlinear equations is close to linear; therefore the well developed analytical methods of the linear control theory can be applied to the problem solution of the nonlinear systems design.

The control at equation (8) is searched also in the quasilinear form:

$$u(\tilde{x}) = -k^T(\tilde{x})\tilde{x} = -\sum_{i=1}^{n} k_i(\tilde{x})\tilde{x}_i; \quad \tilde{x} \in \Omega \in R^n,\ \|\tilde{x}\| \leqslant M_\Omega < \infty. \quad (9)$$

Here $k_i(\tilde{x})$ are some nonlinear functions. Next equation follows from the expressions (8) and (9):

$$\dot{\tilde{x}} = D(\tilde{x})\tilde{x}, \quad (10)$$

where

$$D(\tilde{x}) = A(\tilde{x}) - b(\tilde{x})k^T(\tilde{x}). \quad (11)$$

The characteristic polynomial of the functional matrix $D(\tilde{x})$ (11), in view of the identity $\det(M + bk^T) = \det M + k^T(\mathrm{adj}M)b$ [9], it is possible to present as follows:

$$D(p, \tilde{x}) = \det(pE - D(\tilde{x})) = A(p, \tilde{x}) + b(\tilde{x})\mathrm{adj}(pE - A(\tilde{x}))k^T(\tilde{x})$$

or

$$D(p, \tilde{x}) = A(p, \tilde{x}) + \sum_{i=1}^{n} k_i(\tilde{x})B_i(p, \tilde{x}). \quad (12)$$

Here adj is the adjunct matrix [9, 14] and polynomials are determined by next expressions

$$A(p, \tilde{x}) = \det(pE - A(\tilde{x})) = p^n + \sum_{i=0}^{n-1} \alpha_i(\tilde{x})p^i, \quad (13)$$

$$B_i(p, \tilde{x}) = e_i\mathrm{adj}(pE - A(\tilde{x}))b(\tilde{x}) = \sum_{j=0}^{n-1} \beta_{ij}(\tilde{x})p^j; \quad i = \overline{1, n}, \quad (14)$$

where $e_1 = [1\ 0\ \dots\ 0]$, $e_2 = [0\ 1\ \dots\ 0]$, ..., $e_n = [0\ 0\ \dots\ 1]$.

Let, according to stability and necessary performance of the closed nonlinear system, the desirable characteristic polynomial of the matrix $D(\tilde{x})$ from equation (10) is appointed the following kind:

$$D^*(p) = p^n + \delta_{n-1}^* p^{n-1} + \dots + \delta_1^* p + \delta_0^*. \quad (15)$$

The polynomial (15) satisfies the Gurvits criteria. If the polynomial (15) is to substitute in the equation (12) instead of the polynomial $D(p, \tilde{x})$, a polynomial equation is formed which is equivalent in view of expressions (13), (14) to the following algebraic system:

$$\begin{bmatrix} \beta_{10} & \beta_{20} & \cdots & \beta_{n0} \\ \beta_{11} & \beta_{21} & \cdots & \beta_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1n-1} & \beta_{2n-1} & \cdots & \beta_{n,n-1} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \end{bmatrix}. \quad (16)$$

Here $\eta_i = \eta_i(\tilde{x})$ are the coefficients of the polynomials difference: $D^*(p) - D(p, \tilde{x}) = \eta_0(\tilde{x}) + \eta_1(\tilde{x})p + \ldots + \eta_{n-1}(\tilde{x})p^{n-1}$. In the system (16) the arguments of the functions are lowered for brevity.

Solution of the algebraic system (16) defines the functions $k_i(\tilde{x})$ from control (9) of the closed system (2), (9) or the system (10). The algebraic system (16) has the solution if the next condition is satisfied:

$$\det U(\tilde{x}) = \det[b(\tilde{x})\ A(\tilde{x})b(\tilde{x}) \ldots A^{n-1}(\tilde{x})b(\tilde{x})] \neq 0; \quad \tilde{x} \in \Omega \in R^n, \quad \|\tilde{x}\| \leqslant M_\Omega < \infty. \tag{17}$$

Note, the condition (17) is the controllability condition of the nonlinear plant (8) [9, 10]. If the matrix $A(x) = A$ and the vector $b(x) = b$ in the equation (8) are constants then the inequality (17) passes in the well known Kalman controllability condition.

Thus, if the vector $k(x)$ (9) is determined by the expressions (13)—(16) the constant matrix $D(0)$ and the equilibrium $x \equiv 0$ of the nonlinear closed system (10) are stable. The majority of the control systems designed by this method are asymptotically stable in the bounded domain $\Omega \in R^n$. Only in some cases the equilibrium $x \equiv 0$ of this system is globally asymptotically stable [9]. Hence, if the plant equations are transformed to the quasilinear form, then expressions (9)—(16) allow finding control by which the condition (3) is carried out. Bounded attraction domain Ω_0 from the condition (3) can be found using Lyapunov's function, constructed for stable system $\dot{\tilde{x}} = D(0)\tilde{x}$.

It is easy to see, that expressions (13)—(16) can be applied to design of a modal control for linear plants with constant parameters. But in this case the proposed design method provides the global stability of the closed system. The application of the quasilinear form and the expressions (9)—(16) to control systems design shall be shown on an example below.

Control Systems Design Using JCF of the Plant Equations

Suppose the equations (2) of the plant (1) in the scalar form look like:

$$\dot{\tilde{x}}_i = \varphi_i(\tilde{x}_1, \ldots, \tilde{x}_{i+1}), \; i = \overline{1, n-1}; \tag{18}$$

$$\dot{\tilde{x}}_n = \varphi_n(\tilde{x}_1, \ldots, \tilde{x}_n) + u, \tag{19}$$

where $\varphi_i(\tilde{x}_1, \ldots, \tilde{x}_{i+1}) = f_i(\overline{x}^0_{i+1} + \overline{\tilde{x}}_{i+1}) - f_i(\overline{x}^\circ_{i+1}) = \varphi_i(\overline{\tilde{x}}_{i+1})$, $i = \overline{1, n-1}$ and $f_n(x^\circ + \tilde{x}, u^\circ_0 + u) - f_n(x^\circ, u^\circ_0) = \varphi_n(\tilde{x}) + u$ are differentiable $n - i$ time nonlinear function; $\overline{\tilde{x}}_i = [\tilde{x}_1 \ldots \tilde{x}_i]^T$ is a sub vector and $\overline{\tilde{x}}_n = \tilde{x}$ evidently; $u = u(\tilde{x})$ is the search control.

The controllability conditions of the system (18), (19) can be written down as follows:

$$\left|\frac{\partial\varphi_i(\tilde{x}_1, \ldots, \tilde{x}_{i+1})}{\partial\tilde{x}_{i+1}}\right| \geqslant \varepsilon \neq 0; \quad i = \overline{1, n-1}; \; \tilde{x} \in \Omega \in R^n, \tag{20}$$

where ε there is any positive number. The domain Ω includes the equilibrium $\tilde{x} = 0$.

Definition. If the system of equations (18), (19) satisfies the conditions (20), it is called "Jordan controlled form" [9, 11].

Evidently, the Jordan controlled form is a generalization of the known triangular form of the equations of nonlinear plants [5, 6].

To solve the control system design problem, using the given approach, first of all, the state vector $\tilde{x}$ of the system (18)—(20) is transformed to new state vector

$$w = w(\tilde{x}) = [w_1(\tilde{x}) \quad w_2(\tilde{x}) \quad \ldots \quad w_n(\tilde{x})]^T, \tag{21}$$

where

$$w_1 = \tilde{x}_1, \quad w_i(\overline{\tilde{x}}_i) = \sum_{\nu=1}^{i-1}\frac{\partial w_{i-1}}{\partial\tilde{x}_\nu}\varphi_\nu(\overline{\tilde{x}}_{\nu+1}) + \lambda_{i-1}w_{i-1}(\overline{\tilde{x}}_{i-1}), \quad i = \overline{2, n}, \tag{22}$$

and λ_i are some constants. The transformation $w(\tilde{x})$ (21), (22) is bounded and is convertible by virtue of conditions (20), i.e. in the domain $\Omega \in R^n$ there is a bounded inverse transformation $\tilde{x} = \tilde{x}(w)$ such that $\tilde{x}(w) = \tilde{x}(w(\tilde{x})) = \tilde{x}$.

The stabilizing control $u = u(\tilde{x})$ for plant (18), (19) is determined by the expressions

$$u(\tilde{x}) = -\gamma_1^{-1}(\tilde{x})\left[\gamma_2(\tilde{x}) + \lambda_n w_n(\tilde{x})\right] - \varphi_n(\tilde{x}); \; \tilde{x} \in \Omega; \tag{23}$$

$$\gamma_1(\tilde{x}) = \frac{\partial w_n(\tilde{x})}{\partial\tilde{x}_n} = \prod_{i=1}^{n-1}\frac{\partial\varphi_i(\overline{\tilde{x}}_{i+1})}{\partial\tilde{x}_{i+1}}; \quad \gamma_2(\tilde{x}) = \sum_{\nu=1}^{n-1}\frac{\partial w_n(\tilde{x})}{\partial\tilde{x}_\nu}\tilde{\varphi}_\nu(\overline{\tilde{x}}_{\nu+1}); \tilde{x} \in \Omega, \tag{24}$$

where $\lambda_n \geqslant \varepsilon > 0$; the variables $w_n(\tilde{x})$ are determined by the expressions (22) with $\lambda_i \geqslant \varepsilon > 0$ for all $i = \overline{1, n-1}$ [11—13].

The close system (18)—(20) with control (22)—(24) is described in the new variables $w_i = w_i(\overline{\tilde{x}}_i)$, $i = \overline{1, n}$ by the expressions

$$\dot{w} = \Lambda_n w,\ \Lambda_n = \begin{bmatrix} -\lambda_1 & 1 & \dots & 0 \\ 0 & -\lambda_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & -\lambda_n \end{bmatrix}. \quad (25)$$

Note, the condition (20) ensures the existence of the stabilizing control (23) in the domain $\Omega \in R^n$. The matrix Λ_n (25) coincides with the $n \times n$ Jordan cell [15, p. 142], if $\lambda_i = -\lambda$, $i = \overline{1, n}$. Just therefore the system of equations (18), (19) is called "Jordan controlled form", if condition (20) is carried out in some domain $\Omega \in R^n$. Evidently, the system (25) is asymptotically stable if $\lambda_i \geqslant \varepsilon > 0$, $i = \overline{1, n}$. Since the transformation (21)—(22) is convertible and bounded, then the equilibrium $\tilde{x} = 0$ of the system (18), (19), (22)—(24) with $\lambda_i \geqslant \varepsilon > 0$, $i = \overline{1, n}$ also asymptotically stable in the domain $\Omega \in R^n$.

So, if the nonlinear plant equation (2) is represented in the JCF (18), (19) and conditions (20) are carried out, then the expressions (21)—(24) give the another analytical design method of the nonlinear control systems.

Examples

Example 1. Let, the plant is described in deviations by the equations

$$\dot{\tilde{x}}_1 = \tilde{x}_2, \dot{\tilde{x}}_2 = -a_{21} \sin \tilde{x}_1 + a_{23}\tilde{x}_3, \\ \dot{\tilde{x}}_3 = -\varphi_2(\tilde{x}_2) - \varphi_3(\tilde{x}_3) + u, \quad (26)$$

where $a_{23} \neq 0$ and $\varphi_2(\tilde{x}_2)$, $\varphi_3(\tilde{x}_3)$ are differentiable functions. The state variables $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ are measured. To find the control $u = -k(\tilde{x})\tilde{x}$ by which the equilibrium point $\tilde{x} \equiv 0$ of the plant (26) will be asymptotically stable and duration of the transients does not exceed one second.

In the equations (26) the vector-function $f(0, 0) = 0$, therefore according to the formula (6) the quasilinear equation (8) corresponds to the equations (26) with

$$A(\tilde{x}) = \begin{bmatrix} 0 & 1 & 0 \\ \omega(\tilde{x}_1) & 0 & a_{23} \\ 0 & -a_{32}(\tilde{x}_2) & -a_{33}(\tilde{x}_3) \end{bmatrix},\ b(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (27)$$

where $\omega(\tilde{x}_1) = -a_{21}\tilde{x}_1^{-1} \sin \tilde{x}$, $a_{3j}(\tilde{x}_j) = \tilde{x}_j^{-1}\varphi_j(\tilde{x}_j)$, $j = 2, 3$.

It is easy to see that the equation (8) in view of the expressions (27) is the exact representation of the equations of the nonlinear plant (26). In this case $\det U(\tilde{x}) = -a_{23}^2 \neq 0$, the condition (17) is carried out, i.e. the solution of the design problem exists.

Passing to its definition, we find by the formulas (13) and (14) the polynomials:

$$A(p, \tilde{x}) = p^3 + a_{33}(\tilde{x}_3)p^2 + \alpha_1(\tilde{x})p + \alpha_0(\tilde{x}),$$
$$B_1(p, \tilde{x}) = a_{23}, B_2(p, \tilde{x}) = a_{23}p, B_3(p, \tilde{x}) = p^2 - \omega(\tilde{x}_1),$$

where $\alpha_1(\tilde{x}) = a_{23}a_{32}(\tilde{x}_2) - \omega(\tilde{x}_1)$, $\alpha_0(\tilde{x}) = -a_{33}(\tilde{x}_3)\omega(\tilde{x}_1)$. Let the desirable polynomial

$$D^*(p) = p^3 + \delta_2^* p^2 + \delta_1^* p + \delta_0^* \quad (28)$$

satisfies to the Gurvits criterion. Then the system (17) will become

$$\begin{bmatrix} a_{23} & 0 & -\omega(\tilde{x}_1) \\ 0 & a_{23} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \delta_0^* + a_{33}(\tilde{x}_3)\omega(\tilde{x}_1) \\ \delta_1^* - a_{23}a_{32}(\tilde{x}_2) + \omega(x_1) \\ \delta_2^* - a_{33}(\tilde{x}_3) \end{bmatrix}.$$

The solution of this system are functions: $k_3(\tilde{x}) = \delta_2^* - a_{33}(\tilde{x}_3)$, $k_2(\tilde{x}) = a_{23}^{-1}(\delta_1^* + \omega(\tilde{x}_1)) - a_{32}(\tilde{x}_2)$, $k_1(\tilde{x}) = a_{23}^{-1}(\delta_0^* + \delta_2^*\omega(\tilde{x}_1))$. In according to the expression (9) this solution leads to control:

$$u(\tilde{x}) = -a_{23}^{-1}\delta_0^*\tilde{x}_1 + \delta_2^* \sin \tilde{x}_1 - \\ - a_{23}^{-1}(\delta_1^* + \omega(\tilde{x}_1))\tilde{x}_2 + \varphi_2(\tilde{x}_2) - \delta_2^*\tilde{x}_3 + \varphi_3(\tilde{x}_3). \quad (29)$$

It is easy to establish, that the characteristic polynomial $D(p, x)$, calculated under the expression (11), is equal to the desirable polynomial $D^*(p)$ (28). Hence, the equilibrium point $x \equiv 0$ of the closed system (26), (29) is asymptotically stable in some bounded domain [9].

The closed system was simulated in MATLAB with $\delta_0^* = 81$, $\delta_1^* = 27$, $\delta_2^* = 9$, $\delta_3^* = 1$ and $\tilde{x}_0 = [0{,}25 \quad 3 \quad 0]$. Schedules of the variables of the designed system (26), (29) are shown on Fig. 1.

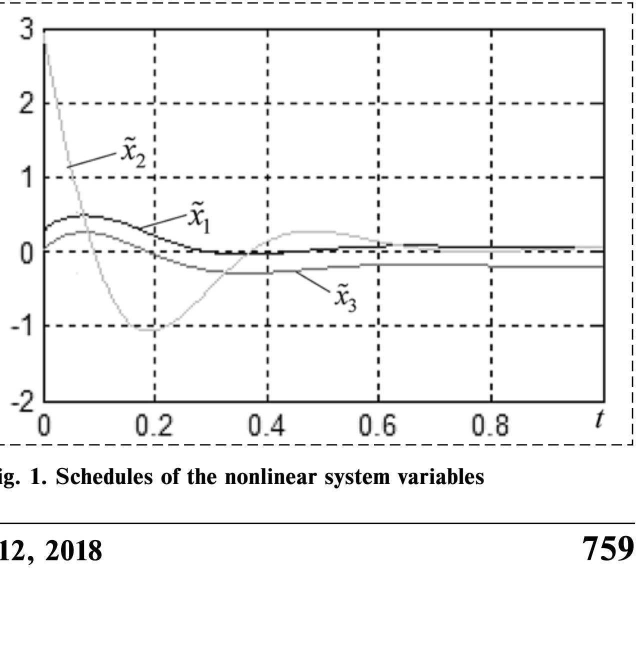

Fig. 1. Schedules of the nonlinear system variables

These schedules allow concluding, that the designed system is asymptotically stable and duration of the transients does not exceed 0.6 second.

Example 2. Suppose, a nonlinear plant is described by the equations

$$\dot{x}_1 = x_2 x_3 + u;\ \dot{x}_2 = x_3;\ \dot{x}_3 = (1 + x_2^2)x_1. \quad (30)$$

Control u must be found by method with application JCF. The equilibrium $x \equiv 0$ of the close system must be asymptotically stable and time response not more then 2.5 second.

In this case $n = 3$ but, form of the equations (30), evidently, does not meet to the JCF of the equations (18), (19). That the equations (30) had this form, we shall designate the state variables so: $x_1 = \tilde{x}_3,\ x_2 = \tilde{x}_1,\ x_3 = \tilde{x}_2$. The resulting equations of the plant (30) look like:

$$\dot{\tilde{x}}_1 = \tilde{x}_2 = \varphi_1(\tilde{x});\ \dot{\tilde{x}}_2 = (1 + \tilde{x}_1^2)\tilde{x}_3 = \varphi_2(\tilde{x});$$
$$\dot{\tilde{x}}_3 = \tilde{x}_1\tilde{x}_2 + u. \quad (31)$$

Equations (31) satisfy the conditions (20), since $\partial\varphi_1(\tilde{x})/\partial\tilde{x}_2 = 1$ and $\partial\varphi_2(\tilde{x})/\partial\tilde{x}_3 = (1 + \tilde{x}_1^2)$ for all $\tilde{x} \in R^3$. Therefore, equations (31) have JCF, and the design task has a solution.

According to the proposed JCF method the transformation (21) is determined by the expressions (22) and (31) have next kind:

$$w_1 = \tilde{x}_1;\ w_2 = \tilde{x}_2 + \lambda_1\tilde{x}_1;$$
$$w_3 = (1 + \tilde{x}_1^2)\tilde{x}_3 + (\lambda_1 + \lambda_2)\tilde{x}_2 + \lambda_1\lambda_2\tilde{x}_1. \quad (32)$$

The transformation (32) is not singular, convertible and bounded for all $\tilde{x} \in R^3$, $\|\tilde{x}\| < \infty$. The functions $\gamma_1(\tilde{x})$ and $\gamma_2(\tilde{x})$ are determined by the expressions (24), (32) and (31) as:

$$\gamma_1(\tilde{x}) = (1 + \tilde{x}_1^2);$$
$$\gamma_2(\tilde{x}) = (2\tilde{x}_1\tilde{x}_3 + \lambda_1\lambda_2)\tilde{x}_2 + (\lambda_1 + \lambda_2)(1 + \tilde{x}_1^2)\tilde{x}_3. \quad (33)$$

Now the search control is written under the expression (23) as:

$$u(\tilde{x}) = -\tilde{x}_1\tilde{x}_2 - (\gamma_2(\tilde{x}) + \lambda_3 w_3(\tilde{x}))/(1 + \tilde{x}_1^2). \quad (34)$$

The expressions for this control in initial designations of the plant variables are obvious and here are not given. Transients of the nonlinear system (31), with controls (34), (32), (33) are submitted on Fig. 2. These schedules are received by simulation of the designed system in MATLAB with $a_{21} = 2$, $a_{23} = 2{,}5$, $\varphi_2(\tilde{x}_2) = 3\,\mathrm{arctg}(\tilde{x}_2)$; $\varphi_3(\tilde{x}_3) = 9\,\mathrm{arctg}(\tilde{x}_3)$, $\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_1 = 7$ and $\tilde{x}_0 = [0{,}3\ \ 0\ \ 3]^T$.

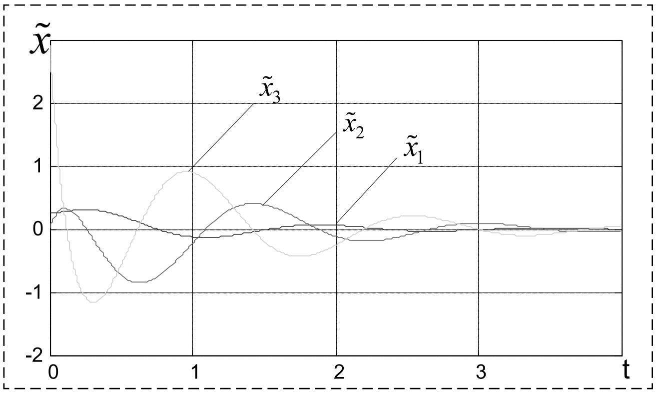

Fig. 2. Schedules of the system state variables

Apparently, the transient's character of the nonlinear control systems can be changed by a choice of the values of the coefficients λ_i, $i = \overline{1,\ n}$.

Example 3. Nonlinear plant is described by the equations

$$\dot{x}_1 = \sin 0{,}1x_1 + b_1 u;\ \dot{x}_2 = u\sqrt{0{,}25a^2 + x_2}, \quad (35)$$

where b_1 and $a > 0$ are constant parameters. To find the control $u(x)$ by which the equilibrium $x \equiv 0$ of the plant (34) will be asymptotically stable.

The form of the equations (35) does not meet to the form of the equations (18), (19). A second order system $\dot{x} = f(x) + b(x)u$ can be transformed to JCF [9], if the following condition is care out:

$$K(x) = [G_{x1}(x)\ \ G_{x2}(x)]b(x) -$$
$$- (b_{1x}(x) + b_{2x}(x))G(x) \neq 0, \quad (36)$$

where $G(x) = \det[f(x)\ \ b(x)]$, $G_{xi}(x) = \partial G(x)/\partial x_i$, $b_{ix}(x) = \partial b_i(x)/\partial x_i$, $i = 1, 2$. In this case the functions $G(x) = b_2(x_2)\sin 0{,}1x_1$, $K(x) = b_1 b_2(x_2)\cos 0{,}1x_1$ and $b_2(x_2) = \sqrt{0{,}25a^2 + x_2}$. Therefore, according to a condition (36), the equations (35) can be transformed to JCF until $|x_1| < 5\pi$ and $-0{,}25a^2 < x_2 \leqslant M < \infty$. Transformation $x_1 = \tilde{x}_1 + b_1\tilde{x}_2$, $x_2 = 0{,}25[(a + \tilde{x}_2)^2 - a^2]$ results the equations (35) in the kind: $\dot{\tilde{x}}_1 = \sin 0{,}1(\tilde{x}_1 + b_1\tilde{x}_2)$, $\dot{\tilde{x}}_2 = u$. These equations have JCF and, using expressions (22)—(25), we find the control

$$u(\tilde{x}) = -\frac{\sin\beta(\tilde{x})}{b_1} - \frac{(\lambda_1 + \lambda_1)\sin\beta(\tilde{x}) + \lambda_1\lambda_1\tilde{x}_1}{0{,}1b_1\cos\beta(\tilde{x})}, \quad (37)$$

where $\beta(\tilde{x}) = 0,1(\tilde{x}_1 + b_1\tilde{x}_2)$, $|\tilde{x}_1 + b_1\tilde{x}_2| < 5\pi$. The attraction domain of the equilibrium is bounded in this case. This domain can be determined by using of the linear character of the control systems (25) in variables w_i, $i = \overline{1,\ n}$ [9].

Conclusion

Representation of the nonlinear plants equations in the quasilinear form or in the Jordan controlled form gives possibility to find analytically the controls as nonlinear feedback on the state variables. These controls can provide asymptotic stability of the system equilibrium in some domain, duration and character of transients. Possibility conditions of transformation of the plant equations to the quasilinear form are very simple: nonlinear functions should be differentiable. Conditions of transformation to the Jordan controlled form are more complex. Generally, these conditions are not found. However representation of the plant equations in the Jordan controlled form is not of a rigid restriction kind because the equations of the many real plants have this form or can be transformed to the JCF form by replacement of the state variables.

References

1. **Isidori A.** Nonlinear control systems (2nd edition), New York, Springer-Verlag, 1989.
2. **Nikiforov V. O.** *Nelineynye sistemy upravleniya c rezhektsiey vneshnikh determinirovannykh vozmuscheniy* (Nonlinear control system with rejection of the external determined disturbances), *Izvestiya RAN. The theory and control systems*, 1997, no. 4, pp. 69—73 (in Russian).
3. **Savić S., Raković M., Penčić M., Borovac B.** Nonlinear motion of humanoid robot upper-body for manipulation task, *Facta Universitatis. Series: Automatic Control and Robotics*, 2014, vol. 13, no. 1, pp. 1—14.
4. **Kim D. P.** *Teoriya avtomaticheskogo upravleniya. Tom 2. Mnogomernye, nelineynye, optimal'nye i adaptivnye sistemy* (Theory of automatic control. Vol. 2. Multivariable, nonlinear, optimal and adaptive systems), Moscow, Phizmatlit, 2004 (in Russian).
5. **Åström K. J., Wittenmark B.** Adaptive control, New York, Addison-Wesley Publishing Company, 1995.
6. **Krstić M., Kanellakopoulos I., Kokotović P. V.** Nonlinear and adaptive control design, New York, John Willey and Sons, 1995.
7. **Lukyanov A. G., Utkin V. I.** Methods of transform of dynamic systems equations to regular form, *Automation and Remote Control*, 1981, no. 4, pp. 5—13.
8. **Egorov I. G.** *K ustoychivosti v tselom nulevogo resheniya sistemy dvukh differentsial'nykh uravneniy* (To stability at whole of the zero solution of two differential equations system), *Differential equations*, 1991, vol. 27, no. 9, pp. 1554—1549 (in Russian).
9. **Gaiduk A. R.** *Teoriya i metody analiticheskogo sinteza sistem avtomaticheskogo upravleniya* (Theory and methods of automatic control systems analytical design), Moscow, Phizmatlit, 2012 (in Russian).
10. **Podchukaev V. A.** *Analiticheskie metody teorii avtomaticheskogo upravleniya* (Analytical methods of the automatic control theory), Moscow, Phizmatlit, 2002 (in Russian).
11. **Gaiduk A. R., Plaksienko E. A., Shapovalov I. O.** Optimal control based on Jordan controlled form, *Proceedings of the 14th International Conference on Circuits, Systems, Electronics, Control & Signal Processing (CSECS '15),* Seljuk University, Konya, Turkey. May 20—22, 2015, pp. 13—18 (CSECS-01).
12. **Gaiduk A. R.** Control systems design with disturbance rejection based on JCF of the nonlinear plant equations, *Facta Universitatis. Series: Automatic Control and Robotics*, 2012, vol. 11, no. 2, pp. 81—90. Available at: facta.junis.ni.ac.rs/acar/acar201202/acar20120201.pdf
13. **Gaiduk A. R., Plaksienko E. A., Kolokolova K. B.** *Sintez algoritmov upravleniya nelineynymi mnogomernymi ob''ektami na osnove UFZH* (Design of control algorithms by nonlinear multivariate objects on basis JCF), *Nauchnyi vestnik NGTU*, 2015, no. 2 (59), pp. 59—72. DOI: 10.17212/1814-1196-2015-2-59-72 (in Russian).
14. **Fikhtengolts M.** *Differentsial'noe i integral'noe ischislenie* (Differential and integral calculus). Vol. 3, Moscow, Nauka, 1969 (in Russian).
15. **Lankaster P.** Theory of matrices, New York, Academic Press, 1969.