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Complete Pole Placement Method for Linear MIMO Systems

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A complete pole placement method for linear MIMO systems with the use of state feedback is presented. The method is based on specific decomposition of representation in the state space of the original MIMO system. The converted representation of the MIMO system contains explicit elements, changing of which with the help of the feedback, enables a specified complete placement of the closed-loop system's poles. The method does not require special solving of matrix equations (like Sylvester equations), which are expressed in the same form for both continuous and discrete cases of the MIMO system description, and does not place restrictions on the algebraic and geometric multiplicity of the specified poles.

Keywords: *decomposition, modal synthesis, linear MIMO-system*

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1. Introduction

The problem of pole placement or eigenvalue assignment for linear dynamic systems with continuous and discrete time has been considered in various formulations in numerous papers (see, for instance, [1]—[13]), yet it has not lost its relevance, especially for Multiple-Input Multiple-Output dynamic systems (MIMO systems).

Let us consider a linear MIMO system of the following form:

$$\mathcal{D}\mathbf{x}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t), \qquad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector; $\mathbf{u} \in \mathbb{R}^r$ is the input vector; $\mathbb{R}$ is a set of real numbers; $n > r$; and $\mathcal{D}$ is a symbol denoting either the differentiation operator, that is, $\mathcal{D}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$, or the shift operator in time $\mathcal{D}\mathbf{x}(t) = \mathbf{x}(t + 1)$.

It is assumed that the matrices $\boldsymbol{B} \in \mathbb{R}^{n \times r}$ and $\boldsymbol{C} \in \mathbb{R}^{m \times n}$ are full rank matrices, and the matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has a set of eigenvalues

$$\text{eig}(\boldsymbol{A}) = \{\lambda_i \in \mathbb{C}: \det(\lambda \boldsymbol{I}_n - \boldsymbol{A}) = 0\},$$

where $\boldsymbol{I}_n$ is the identity matrix of size $n \times n$; $\mathbb{C}$ is the set of complex numbers (complex plane) and necessarily includes $\lambda_i \in \mathbb{C}$ such that $\text{Re}(\lambda_i) > 0$ for $\mathcal{D}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$ and $|\lambda_i| > 1$ for $\mathcal{D}\mathbf{x}(t) = \mathbf{x}(t + 1)$. Here, $|\lambda_i|$ is the absolute value of λ_i.

Let us introduce the concept of $\mathbb{C}^{\text{stab}}$, which, depending on the type of MIMO system under study (continuous or discrete), denotes either the left half $\mathbb{C}^-$ of the plane, that is, $\mathbb{C}^{\text{stab}} \doteq \mathbb{C}^-$, or the interior of the unit circle centered at the origin of $\mathbb{C}$, that is, $\mathbb{C}^{\text{stab}} \doteq \mathbb{C}_{|\lambda| < 1}$.

It is assumed that for the MIMO system (1.1), there exists feedback control of the form

$$\mathbf{u}(t) = \boldsymbol{F}\mathbf{x}(t), \qquad (2)$$

where $\boldsymbol{F} \in \mathbb{R}^{r \times n}$ is the state controller matrix.

The control of system (1) using laws (2) is the classical problem in which it is necessary to find a matrix $\boldsymbol{F}$ such that certain prescribed requirements for the control process are fulfilled. These requirements can be divided into three groups [7]:

a) requirements for the pole placement of the closed-loop system (eigenvalues of the matrix $\boldsymbol{A} + \boldsymbol{BF}$) at the prescribed points of $\mathbb{C}^{\text{stab}}$ or in the prescribed domain $\mathbb{C}^{\text{stab}}$;

b) requirements for placement of poles and zeros (certain zeros of the transfer matrix of the MIMO system [13], [14]) of the closed-loop system at the prescribed points of $\mathbb{C}^{\text{stab}}$ or in the prescribed domains of $\mathbb{C}^{\text{stab}}$;

c) requirements for the transient processes in the closed-loop system in the sense of the minimum of a certain functional.

Requirement (a) applies to all known formulations of the stabilization problem. In this case, additional conditions of complete controllability and complete observability of the system are typically imposed.

Requirement (a) is especially pronounced in formulations of modal control [1], [2], [4]—[8], [12]—[17].

It is well known that the characteristic polynomial

$$\det(\lambda \boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{BF}), \qquad (3)$$

where $\lambda = s$ for the case of $\mathcal{D}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$ and $\lambda = z$ for the case of $\mathcal{D}\mathbf{x}(t) = \mathbf{x}(t + 1)$, determines the pole location of the closed-loop system on $\mathbb{C}$, which determines the stability of MIMO system (1). Imposing the requirements for the desirable (in the sense of condition (a)) location of poles, the stability and (indirectly) the quality of transient processes in the closed-loop system can be ensured.

The requirements for the pole location can be specified by factorizing polynomial (3); for example,

$$\det(\lambda \boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{BF}) = (\lambda - \tilde{\lambda}_1)(\lambda - \tilde{\lambda}_2)...(\lambda - \tilde{\lambda}_n),(4)$$

where $\tilde{\lambda}_i$ are the prescribed values of the polynomial roots (eigenvalues of the matrix $\boldsymbol{A} + \boldsymbol{BF}$) or of the decomposition of the matrix

$$\boldsymbol{A} + \boldsymbol{BF} = \boldsymbol{W}\Lambda \boldsymbol{W}^{-1}, \quad (5)$$

where Λ is a block-diagonal matrix and W is the transformation matrix.

In the matrix Λ, for each i-th real pole λ_i corresponding to the given value of the root of the characteristic polynomial (4), there exists a block with a size of 1×1, and for each pair of complex conjugate roots, there exists a block with a size of 2×2 of the form

$$\begin{pmatrix} \mathrm{Re}(\lambda_i) & \mathrm{Im}(\lambda_i) \\ -\mathrm{Im}(\lambda_i) & \mathrm{Re}(\lambda_i) \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

If multiple roots are given, this is reflected in the structure of the matrix Λ, similarly to the Jordan form of the matrix [7, 8].

Another method of fulfilling requirement (a) is to use linear matrix inequality (LMI) domains [1], [11]. Let D be a given convex domain $\mathbb{C}^{stab}$ in the sense of requirement (a); then, there exist LMIs describing the boundaries of this domain.

For MIMO systems with $n \gg 1$, the available methods of pole placement are often inapplicable in practice because of their disadvantages, such as ill-conditioned matrices (for example, controllability matrices), possible insolvability of the problem in the case of complete controllability (for example, the constraint of the form of the difference of the algebraic and geometric multiplicities of the assigned poles), fast growth of dimensionality of the equations to be solved, and so on.

This paper presents a complete pole placement method in the MIMO system with state feedback; that is, in this method, requirement (a) for the MIMO system (1) is ensured using law (2) in the sense of placing the eigenvalues of the matrix $\boldsymbol{A} + \boldsymbol{BF}$ in the domain $\mathbb{C}^{stab}$. The method is based on a special similarity transformation of the original system. The elements of the matrix $\boldsymbol{A}$ and (or) their combinations are determined explicitly, and the change of these elements using feedback makes it possible to ensure the stability of the closed-loop system. As will be shown later, this method does not require the solution of any special matrix equations (like Sylvester's equation), has the same form for the continuous and discrete system models, has no limitations with respect to the algebraic and geometric multiplicities of the poles, and can be easily used for the synthesis of regulators in systems with a large dimension of the state space.

2. Decomposition of MIMO systems

Let $\boldsymbol{B}^{\perp}$ be a rectangular matrix, called the divisor of zero, that satisfies the conditions [8], [18]

$$\boldsymbol{B}^{\perp}\boldsymbol{B} = \boldsymbol{0}_{r \times r}, \quad (6)$$

$$\boldsymbol{B}^{\perp}\boldsymbol{B}^{\perp +} = \boldsymbol{I}_{n-r}, \quad (7)$$

where $\boldsymbol{B}^{+}$ and $\boldsymbol{B}^{\perp +}$ are pseudoinverse Moore-Penrose matrices; $\boldsymbol{0}_{r \times r}$ is the zero matrix of size $r \times r$. We take a non-singular matrix [18]

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{B}^{\perp} \\ \boldsymbol{B}^{+} \end{pmatrix}, \quad (8)$$

which has the inverse matrix:

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \boldsymbol{B}^{\perp} \\ \boldsymbol{B}^{+} \end{pmatrix}^{-1} = (\boldsymbol{B}^{\perp +} \,\vdots\, \boldsymbol{B}). \quad (9)$$

Performing multiplication of the initial and inverse matrices, we obtain the identity

$$(\boldsymbol{B}^{\perp +} \,\vdots\, \boldsymbol{B})\begin{pmatrix} \boldsymbol{B}^{\perp} \\ \boldsymbol{B}^{+} \end{pmatrix} = \boldsymbol{B}^{\perp +}\boldsymbol{B}^{\perp} + \boldsymbol{B}\boldsymbol{B}^{+} = \boldsymbol{I}_n, \quad (10)$$

$$\begin{pmatrix} \boldsymbol{B}^{\perp} \\ \boldsymbol{B}^{+} \end{pmatrix}(\boldsymbol{B}^{\perp +} \,\vdots\, \boldsymbol{B}) = \begin{pmatrix} \boldsymbol{I}_{n-r} & \boldsymbol{0}_{r \times r} \\ \boldsymbol{0}_{r \times (n-r)} & \boldsymbol{I}_r \end{pmatrix}. \quad (11)$$

Note that the operations "⊥" and "+" commute for the matrices of full rank; that is, in this case

$$\boldsymbol{B}^{\perp +} = \boldsymbol{B}^{+\perp}. \quad (12)$$

If the selected matrix $\boldsymbol{B}^{\perp}$ is the orthogonal matrix, then the condition (2.2) will transform to the following:

$$\boldsymbol{B}^{\perp}\boldsymbol{B}^{\perp \mathrm{T}} = \boldsymbol{I}_{n-r}. \quad (13)$$

Consider the multilevel decomposition of the MIMO system (1.1) with the matrices ($\boldsymbol{A}$, $\boldsymbol{B}$), where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$.

Zero (initial) level

$$\boldsymbol{A}_0 = \boldsymbol{A},\ \boldsymbol{B}_0 = \boldsymbol{B}. \quad (14)$$

First level

$$\boldsymbol{A}_1 = \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp +},\ \boldsymbol{B}_1 = \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B} \quad (15)$$

k-th (intermediate) level

$$\boldsymbol{A}_k = \boldsymbol{B}_{k-1}^{\perp}\boldsymbol{A}_{k-1}\boldsymbol{B}_{k-1}^{\perp +},\ \boldsymbol{B}_k = \boldsymbol{B}_{k-1}^{\perp}\boldsymbol{A}_{k-1}\boldsymbol{B}_{k-1} \quad (16)$$

L-th (final) level, $L = \mathrm{ceil}(n/r) - 1$,

$$\boldsymbol{A}_L = \boldsymbol{B}_{L-1}^{\perp}\boldsymbol{A}_{L-1}\boldsymbol{B}_{L-1}^{\perp +},\ \boldsymbol{B}_L = \boldsymbol{B}_{L-1}^{\perp}\boldsymbol{A}_{L-1}\boldsymbol{B}_{L-1}. \quad (17)$$

Here ceil(*) is the operation of rounding the number (*) upwards; for example, ceil(0.1) = 1, ceil(1.6) = 2, ceil(2.01) = 3, and so on.

Theorem 1. If the MIMO system (1.1) with a pair of matrices $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable, the all pairs of matrices $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ (2.10)—(2.12) are also completely controllable.

Proof of Theorem 1. It is known that, for complete control of MIMO systems, it is necessary and sufficient that [1, 2, 7, 8, 11, 14]

$$\forall\lambda \in \mathbb{C}: \operatorname{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}_n \,\vdots\, \boldsymbol{B}) = n. \tag{18}$$

The condition $\forall\lambda \in \mathbb{C}$ can be replaced by the condition $\forall\lambda \in \operatorname{eig}(\boldsymbol{A})$.

Using matrix (2.3), we transform the pencil of matrices as:

$$\boldsymbol{T}(\boldsymbol{A} - \lambda \boldsymbol{I}_n \,\vdots\, \boldsymbol{B}) = \begin{pmatrix} \boldsymbol{B}^{\perp} \\ \cdots \\ \boldsymbol{B}^{+} \end{pmatrix} (\boldsymbol{A} - \lambda \boldsymbol{I}_n \,\vdots\, \boldsymbol{B}). \tag{19}$$

Expanding the right-hand side of (2.14), we obtain

$$\begin{pmatrix} \boldsymbol{B}^{\perp} \\ \cdots \\ \boldsymbol{B}^{+} \end{pmatrix} (\boldsymbol{A} - \lambda \boldsymbol{I}_n \,\vdots\, \boldsymbol{B}) = \begin{pmatrix} \boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) & \boldsymbol{0}_{(n-r)\times r} \\ \boldsymbol{B}^{+}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) & \boldsymbol{I}_r \end{pmatrix},$$

and furthermore, due to the nonsingularity of the matrix T (2.3), we have

$$\operatorname{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}_n \,\vdots\, \boldsymbol{B}) = \operatorname{rank}\begin{pmatrix} \boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) & \boldsymbol{0}_{(n-r)\times r} \\ \boldsymbol{B}^{+}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) & \boldsymbol{I}_r \end{pmatrix}. \tag{20}$$

It follows from the structure of (2.15) that the submatrix

$$(\boldsymbol{B}^{+}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) \,\vdots\, \boldsymbol{I}_r)$$

has the rank r for all λ. Therefore, for the condition (2.13) to be satisfied, it is necessary and sufficient that the rank of the submatrix $\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n)$ satisfy the condition

$$\forall\lambda \in \mathbb{C}: \operatorname{rank}\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = n - r.$$

We carry out the nonsingular transformation of the submatrix $\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n)$ as

$$\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{T}^{-1} = \boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n)(\boldsymbol{B}^{\perp\mathrm{T}} \,\vdots\, \boldsymbol{B}). \tag{21}$$

Expanding the right side of (2.16),

$$\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n)(\boldsymbol{B}^{\perp\mathrm{T}} \,\vdots\, \boldsymbol{B}) =$$

$$= (\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}} - \lambda \boldsymbol{I}_{n-r} \,\vdots\, \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}) = (\boldsymbol{A}_1 - \lambda \boldsymbol{I}_{n-r} \,\vdots\, \boldsymbol{B}_1).$$

Then, similarly to the previous case (2.15), we have

$$\operatorname{rank}\boldsymbol{B}^{\perp}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = \operatorname{rank}(\boldsymbol{A}_1 - \lambda \boldsymbol{I}_{n-r} \,\vdots\, \boldsymbol{B}_1). \tag{22}$$

Comparing the right-hand sides of (2.13) and (2.17), we arrive at the following intermediate result: the MIMO system (1.1) is completely controllable if and only if the pair of matrices $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is completely controllable.

Then, transforming the pair of matrices $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ similarly to how this was done above, we obtain by induction the assertion of Theorem 1. The proof of Theorem 1 is complete.

3. Synthesis of the MIMO system regulator and parameterization

Without loss of generality, we assume that all the matrices $\boldsymbol{B}_i$ in (2.9)—(2.12) are the matrices of full rank [8], [16]. Then the following statement is true:

Theorem 2. Let the MIMO system (1.1) be completely controllable, and let matrix $\boldsymbol{F} \in \mathbb{R}^{r\times m}$ satisfy the conditions:

$$\boldsymbol{F} = \boldsymbol{F}_0 = \Phi_0 \boldsymbol{B}_0^{-} - \boldsymbol{B}_0^{-}\boldsymbol{A},\ \boldsymbol{B}_0^{-} = \boldsymbol{B}_0^{+} - \boldsymbol{F}_1 \boldsymbol{B}_0^{\perp}, \tag{23}$$

$$\boldsymbol{F}_1 = \Phi_1 \boldsymbol{B}_1^{-} - \boldsymbol{B}_1^{-}\boldsymbol{A}_1,\ \boldsymbol{B}_1^{-} = \boldsymbol{B}_1^{+} - \boldsymbol{F}_2 \boldsymbol{B}_1^{\perp}, \ldots \tag{24}$$

$$\boldsymbol{F}_k = \Phi_k \boldsymbol{B}_k^{-} - \boldsymbol{B}_k^{-}\boldsymbol{A}_k,\ \boldsymbol{B}_k^{-} = \boldsymbol{B}_k^{+} - \boldsymbol{F}_{k+1} \boldsymbol{B}_k^{\perp}, \ldots \tag{25}$$

$$\boldsymbol{F}_L = \Phi_L \boldsymbol{B}_L^{+} - \boldsymbol{B}_L^{+}\boldsymbol{A}_L. \tag{26}$$

Then

$$\operatorname{eig}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}) = \bigcup_{i=1}^{L+1} \operatorname{eig}(\Phi_{i-1}). \tag{27}$$

P r o o f o f T h e o r e m 2. Consider the following formulas for the controller matrix:

$$\boldsymbol{F} = \Phi \boldsymbol{B}^{-} - \boldsymbol{B}^{-}\boldsymbol{A},\ \boldsymbol{B}^{-} = \boldsymbol{B}^{+}(\boldsymbol{I}_n - \boldsymbol{B}\boldsymbol{F}_1\boldsymbol{B}^{\perp}).$$

Then, we have the chain of nonsingular (similarity) transformations

$$\begin{pmatrix} \boldsymbol{B}^{\perp} \\ \cdots \\ \boldsymbol{B}^{+} \end{pmatrix} (\boldsymbol{A} + \boldsymbol{B}(\Phi\boldsymbol{B}^{-} - \boldsymbol{B}^{-}\boldsymbol{A}))(\boldsymbol{B}^{\perp+} \,\vdots\, \boldsymbol{B}) =$$

$$= \begin{pmatrix} \boldsymbol{B}^{\perp}\boldsymbol{A} \\ \cdots \\ \Phi\boldsymbol{B}^{+} - \Phi\boldsymbol{F}_1\boldsymbol{B}^{\perp} + \boldsymbol{F}_1\boldsymbol{B}^{\perp}\boldsymbol{A} \end{pmatrix} (\boldsymbol{B}^{\perp+} \,\vdots\, \boldsymbol{B}) =$$

$$= \begin{pmatrix} \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp+} & \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B} \\ \boldsymbol{F}_1\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp+} - \Phi\boldsymbol{F}_1 & \Phi + \boldsymbol{F}_1\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B} \end{pmatrix}.$$

To the resulting matrix

$$\begin{pmatrix} \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp+} & \boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B} \\ \boldsymbol{F}_1\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp+} - \Phi\boldsymbol{F}_1 & \Phi + \boldsymbol{F}_1\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B} \end{pmatrix} \tag{28}$$

we apply the nonsingular similarity transformation:

$$\left(\begin{array}{c:c} I_{n-m} & 0 \\ \hdashline -F_1 & I_m \end{array}\right) = \left(\begin{array}{c:c} I_{n-m} & 0 \\ \hdashline F_1 & I_m \end{array}\right)^{-1}. \qquad (29)$$

If (3.7) is multiplied on the left by (3.6), we obtain

$$\left(\begin{array}{c:c} I_{n-m} & 0 \\ \hdashline -F_1 & I_m \end{array}\right)\left(\begin{array}{c:c} B^{\perp}AB^{\perp+} & B^{\perp}AB \\ \hdashline F_1B^{\perp}AB^{\perp+} - \Phi F_1 & \Phi + F_1B^{\perp}AB \end{array}\right) =$$

$$= \left(\begin{array}{c:c} B^{\perp}AB^{\perp+} & B^{\perp}AB \\ \hdashline -\Phi F_1 & \Phi \end{array}\right).$$

The multiplication of the result of the preceding transformation on the right by the inverse of (3.7) yields

$$\left(\begin{array}{c:c} B^{\perp}AB^{\perp+} & B^{\perp}AB \\ \hdashline -\Phi F_1 & \Phi \end{array}\right)\left(\begin{array}{c:c} I_{n-m} & 0 \\ \hdashline -F_1 & I_m \end{array}\right) =$$

$$= \left(\begin{array}{c:c} B^{\perp}AB^{\perp+} + B^{\perp}ABF_1 & B^{\perp}AB \\ \hdashline 0 & \Phi \end{array}\right).$$

Thus, the nonsingular similarity transformation (3.7) is used to obtain the matrix

$$\left(\begin{array}{c:c} B^{\perp}AB^{\perp+} + B^{\perp}ABF_1 & B^{\perp}AB \\ \hdashline 0 & \Phi \end{array}\right) = \left(\begin{array}{c:c} A_1 + B_1F_1 & B_1 \\ \hdashline 0 & \Phi \end{array}\right). \qquad (30)$$

Its eigenvalues clearly have the form:

$$\mathrm{eig}\left(\begin{array}{c:c} B^{\perp}AB^{\perp+} + B^{\perp}ABF_1 & B^{\perp}AB \\ \hdashline 0 & \Phi \end{array}\right) =$$

$$= \mathrm{eig}\left(\begin{array}{c:c} A_1 + B_1F_1 & B_1 \\ \hdashline 0 & \Phi \end{array}\right) = \mathrm{eig}(\Phi) \cup \mathrm{eig}(B^{\perp}AB^{\perp+} +$$

$$+ B^{\perp}ABF_1) = \mathrm{eig}(\Phi) \cup \mathrm{eig}(A_1 + B_1F_1).$$

Transforming matrix (3.8) similarly to the above transformations, we obtain

$$\mathrm{eig}\left(\begin{array}{c:c} B_1^{\perp}A_1B_1^{\perp+} + B_1^{\perp}A_1B_1F_2 & B_1^{\perp}A_1B_1 \\ \hdashline 0 & \Phi_1 \end{array}\right) =$$

$$= \mathrm{eig}(\Phi_1) \cup \mathrm{eig}(B_1^{\perp}A_1B_1^{\perp+} + B_1^{\perp}A_1B_1F_2) =$$

$$= \mathrm{eig}(\Phi_1) \cup \mathrm{eig}(A_2 + B_2F_2).$$

If these transformations are continued until the pair of matrices (A_L, B_L), where $L = \mathrm{ceil}(n/r) - 1$, we obtain the equality (3.5), which proves the theorem.

The following controller synthesis algorithm ensuring the prescribed pole placement follows from Theorem 2:

(1) Set the matrices $A_0 = A$ and $B_0 = B$.

(2) Calculate $L = \mathrm{ceil}(n/r) - 1$.

(3) Set the matrices $\Phi = \Phi_0, \Phi_1, ..., \Phi_L$ such that $\bigcup_{i=1}^{L+1} \mathrm{eig}(\Phi_{i-1})$ is the desirable spectrum of the closed-loop system.

(4) Calculate the zero divisor $B_0^{\perp} = B^{\perp}$ and then the pseudoinverse matrices $A_1 = B^{\perp}AB^{\perp+}$, $B_1 = B^{\perp}AB$, ...

(5) Calculate the zero divisor $B_k^{\perp}$, the pseudoinverse matrix $B_k^{\perp+}$, and then the matrices $A_{k+1} = B_k^{\perp}A_kB_k^{\perp+}$, $B_{k+1} = B_k^{\perp}A_kB_k$, ...

(6) Calculate the zero divisor $B_{L-2}^{\perp}$, the pseudoinverse matrices $B_{L-2}^{\perp}$ and $B_k^{\perp+}$, and then the matrices $A_{L-1} = B_{L-2}^{\perp}A_{L-2}B_{L-2}^{\perp+}$ and $B_{L-1} = B_{L-2}^{\perp}A_{L-2}B_{L-2}$.

(7) Calculate the zero divisor $B_{L-1}^{\perp}$, the pseudoinverse matrices $B_{L-1}^{\perp+}$ and $B_k^{\perp+}$, and then the matrices $A_L = B_{L-1}^{\perp}A_{L-1}B_{L-1}^{\perp+}$ and $B_L = B_{L-1}^{\perp}A_{L-1}B_{L-1}$.

(8) Sequentially calculate the matrices:

$$F_L = \Phi_L B_L^{+} - B_L^{+}A_L,$$

$$B_{L-1}^{-} = B_{L-1}^{+} - F_L B_{L-1}^{\perp},$$

$$F_{L-1} = \Phi_{L-1}B_{L-1}^{-}A_{L-1}, ...$$

$$B_1^{-} = B_1^{+} - F_2B_1^{\perp},\ F_1 = \Phi_1B_1^{-} - B_1^{-}A_1,$$

$$B_0^{-} = B_0^{+} - F_1B_0^{\perp},\ F = F_0 = \Phi_0B_0^{-} - B_0^{-}A_0.$$

The controller with matrix (3.1) guarantees that condition (3.5) is fulfilled. A block diagram of the pole placement algorithm for the orthogonal zero divisor is shown in Fig. 1.

Corresponding changes should also be made in the formulas (3.1)—(3.3) of the controller.

The algorithm of the complete pole placement in certain cases provides an opportunity to obtain the final formula of the controller (Ackermann's formula) for the Single Input Multiple Output (SIMO) systems [8]—[10], [14]. For the simplest case of $n = 2r$ ($L = 1$) and the orthogonal divisor of zero, from Theorem 2 we obtain the following final formula of the controller:

$$F = \Phi_0(B^{+} - [\Phi_1(B^{\perp}AB)^{+} - (B^{\perp}AB)^{+}B^{\perp}AB^{\perp\mathrm{T}}]B^{\perp}) -$$
$$- (B^{+} - [\Phi_1(B^{\perp}AB)^{+} - (B^{\perp}AB)^{+}B^{\perp}AB^{\perp\mathrm{T}}]B^{\perp})A,$$

ensuring equality for the set of eigenvalues

$$\mathrm{eig}(A + BF) = \mathrm{eig}(\Phi_0) \cup \mathrm{eig}(\Phi_1).$$

Theorem 2 and formulas (3.1)—(3.5) show that no restrictions are imposed on the matrices Φ_i. Any matrices that satisfy the condition of matching the eigenvalues set with the specified set can be used as the ob-

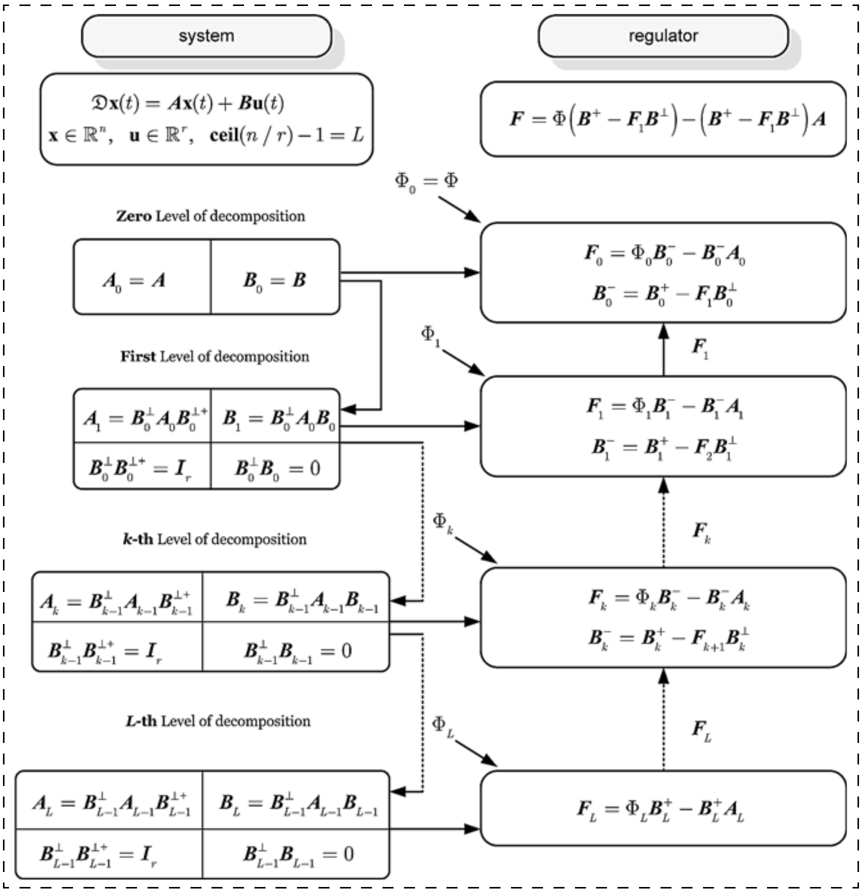

Fig. 1. Block diagram of the pole placement algorithm

jects. In this case, all the matrices Φ_i satisfying the condition

$$\bigcup_{i=1}^{L+1} \mathrm{eig}(\Phi_{i-1}) = \Lambda_{\mathrm{specified}}, \tag{31}$$

where $\Lambda_{\mathrm{specified}}$ is the given set of eigenvalues (specified poles), form a set of the equivalent controllers.

It is not difficult to modify the above algorithm by using the orthogonal matrices instead of the non-orthogonal zero divisors [17].

4. Examples of solutions to the complete pole placement problem

Consider controller synthesis examples that provide complete pole placement.

Example 1. Consider a fully controlled MIMO system for the case of $\mathcal{D}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$ with the matrices [19]

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \ \boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 2}. \tag{32}$$

A feature of the MIMO system with a pair of matrices (4.1) is the difficulty of solving the pole placement problem due to defects in the matrix $\boldsymbol{A}$. This refers to the modifications of the Ackermann and Bass-Gura methods [2], [5], [8].

Also, it is impossible to solve this problem, for example by using the Kautsky-Nichols-Van Dooren method [15] in this situation, if the data for all specified eigenvalues (specified poles) coincide.

Let us show that these difficulties do not affect the workability of the method presented. We make use of the algorithm described in Section 3, executing it incrementally.

1) Define a template for the zero level of decomposition

$$\boldsymbol{A}_0 = \boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\boldsymbol{B}_0 = \boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

2) Calculate the parameter $L = \mathrm{ceil}(3/2) - 1 = 2 - 1 = 1$. It follows from this that for the MIMO system (4.1) there are only two levels of decomposition: the zeroth one and the first one (which is the finite level at the same time).

3) Define matrices with the desirable eigenvalues:

$$\Phi = \Phi_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \Phi_1 = \lambda_3, \tag{34}$$

taking into consideration that $\mathrm{eig}(\Phi_0) \cup \mathrm{eig}(\Phi_1) = \{\lambda_1, \lambda_2, \lambda_3\}$ are specified poles.

4) Calculate the zero divisor $\boldsymbol{B}_0^\perp = \boldsymbol{B}^\perp$, the pseudoinverse matrix, and after that the matrices $\boldsymbol{A}_1 = \boldsymbol{B}_0^\perp \boldsymbol{A}\boldsymbol{B}^{\perp+}$ and $\boldsymbol{B}_1 = \boldsymbol{B}_0^\perp \boldsymbol{A}\boldsymbol{B}^\perp$. We will obtain

$$\boldsymbol{B}_0^\perp = (-1 \ \vdots \ 1 \ \vdots \ 3), \tag{35}$$

$$\boldsymbol{B}_0^{\perp+} = \frac{1}{11}\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \tag{36}$$

......

$$\boldsymbol{A}_1 = \boldsymbol{B}_0^\perp \boldsymbol{A}_0 \boldsymbol{B}_0^{\perp+} =$$

$$= \frac{1}{11}\cdot(-1 \ \vdots \ 1 \ \vdots \ 3)\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \frac{21}{11}, \tag{37}$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_0^\perp \boldsymbol{A}_0 \boldsymbol{B}_0 =$$

$$= (-1 \ \vdots \ 1 \ \vdots \ 3)\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (-1 \ \vdots \ 0). \tag{38}$$

Therefore, at the first decomposition level we have the single-output (that is, the state) and two-input MIMO system.

$$A_1 = B_0^{\perp} A_0 B_0^{\perp +} = \frac{21}{11},$$

$$B_1 = B_0^{\perp} A_0 B_0 = (-1 \,\vdots\, 0). \qquad (39)$$

5) Now it is necessary to calculate the matrices $F_1 = \Phi_1 B_1^{-} - B_1^{-} A_1$, $B_0^{-} = B_0^{+} - F_1 B_0^{\perp}$, and $F_0 = \Phi_0 B_0^{-} - B_0^{-} A_0$ in consecutive order. For this purpose, we define the pseudoinverse matrices

$$B_0^{+} = \frac{1}{11} \cdot \begin{pmatrix} 1 & 10 & -3 \\ 3 & -3 & 2 \end{pmatrix}, \quad B_1^{+} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \qquad (40)$$

Now we can obtain expressions for the corresponding controllers:

$$F_1 = \Phi_1 B_1^{+} - B_1^{+} A_1 + B_1^{\perp} \varpi =$$

$$= \lambda_3 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \frac{21}{11} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \varpi = \begin{pmatrix} -\lambda_3 + \frac{21}{11} \\ \varpi \end{pmatrix}, \qquad (41)$$

$$B^{-} = B_0^{+} - F_1 B_0^{\perp} =$$

$$= \frac{1}{11} \cdot \begin{pmatrix} 1 & 10 & -3 \\ 3 & -3 & 2 \end{pmatrix} - \begin{pmatrix} -\lambda_3 + \frac{21}{11} \\ \varpi \end{pmatrix} (-1 \,\vdots\, 1 \,\vdots\, 3) =$$

$$= \begin{pmatrix} 2 - \lambda_3 & \lambda_3 - 1 & 3\lambda_3 - 6 \\ \frac{3}{11} + \varpi & -\frac{3}{11} - \varpi & \frac{2}{11} - 3\varpi \end{pmatrix}, \qquad (42)$$

$$F_0 = \Phi_0 B_0^{-} - B_0^{-} A_0 =$$

$$= -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 2 - \lambda_3 & \lambda_3 - 1 & 3\lambda_3 - 6 \\ \frac{3}{11} + \varpi & -\frac{3}{11} - \varpi & \frac{2}{11} - 3\varpi \end{pmatrix} -$$

$$- \begin{pmatrix} 2 - \lambda_3 & \lambda_3 - 1 & 3\lambda_3 - 6 \\ \frac{3}{11} + \varpi & -\frac{3}{11} - \varpi & \frac{2}{11} - 3\varpi \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1(2 - \lambda_3) + 2\lambda_3 - 4 & \lambda_1\lambda_3 - \lambda_1 - \lambda_3 & 3\lambda_1\lambda_3 - 6\lambda_1 + 6\lambda_3 + 12 \\ \left(\frac{3}{11} + \varpi\right)(\lambda_2 - 2) & -\left(\frac{3}{11} + \varpi\right)(\lambda_2 + 1) & \left(\frac{2}{11} - 3\varpi\right)(\lambda_2 - 2) \end{pmatrix}. \qquad (43)$$

In formulas (4.10)—(4.12), freedom in specification of the controller for the first level is considered by means of the component $B_1^{\perp}\varpi$, where ϖ is an arbitrary scalar parameter. Indeed, completing the transformation, we obtain

$$A_1 + B_1 F_1 = A_1 + B_1(\Phi_1 B_1^{+} - B_1^{+} A_1 + B_1^{\perp} \varpi) =$$

$$= A_1 + B_1(\Phi_1 B_1^{+} - B_1^{+} A_1) + \underbrace{B_1 B_1^{\perp}}_{0} \varpi.$$

After that, without loss of generality, we assume that in (4.12) there is an arbitrary parameter equal to $\varpi = \omega/11$, where ω is also an arbitrary parameter; then (43) can be rewritten in the equivalent form:

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/11 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \lambda_1(2 - \lambda_3) + 2\lambda_3 - 4 & \lambda_1\lambda_3 - \lambda_1 - \lambda_3 & 3\lambda_1\lambda_3 - 6\lambda_1 + 6\lambda_3 + 12 \\ (3 + \omega)(\lambda_2 - 2) & -(3 + \omega)(\lambda_2 + 1) & (2 - 3\omega)(\lambda_2 - 2) \end{pmatrix}. \qquad (44)$$

It is obvious that the given solution does not contain restrictions on specifying the same pole with a multiplicity of three. Assuming that $\lambda = \lambda_1 = \lambda_2 = \lambda_3$, we obtain

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/11 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \lambda_1(2 - \lambda) + 2\lambda - 4 & \lambda^2 - 2\lambda & 3\lambda^2 + 12 \\ (3 + \omega)(\lambda - 2) & -(3 + \omega)(\lambda + 1) & (2 - 3\omega)(\lambda - 2) \end{pmatrix}. \qquad (45)$$

Here, if the value $\omega = -3$, this would minimize the information used in feedback channels, namely

$$F_0 = \begin{pmatrix} -\lambda^2 + 4\lambda - 4 & \lambda^2 - 2\lambda & 3\lambda^2 + 12 \\ 0 & 0 & \lambda - 2 \end{pmatrix}. \qquad (46)$$

Example 2. Consider a MIMO system (1.1) with discrete time ($\mathcal{D}\mathbf{x}(t) = \mathbf{x}(t + 1)$) and the problem of controller synthesis that ensures a finite duration of the closed system transition process. In this case, the $A + BF$ matrix will only have nulls as its eigenvalues [8]. This requirement means that any nilpotent matrices with an index of nilpotency of no more than r [8], [18] can be taken as the matrices Φ_i.

Choose, for simplicity, zero matrices $\mathbf{0}_{r \times r}$ as the matrices Φ_i. Then the final part of the algorithm that has been discussed in Section 3 of this paper will have the form

$$F_L = -B_L^{-} A_L,$$

$$B_{L-1}^{-} = B_{L-1}^{+} - F_L B_{L-1}^{\perp}, \; F_{L-1} = -B_{L-1}^{-} A_{L-1}, \ldots,$$

$$B_1^{-} = B_1^{+} - F_2 B_1^{\perp}, \; F_1 = -B_1^{-} A_1,$$

$$B_0^{-} = B_0^{+} - F_1 B_0^{\perp}, \; F = F_0 = -B_0^{-} A_0.$$

Here, in the situation where $n = 2r$ ($L = 1$), the formula of the controller that ensures a finite duration of the transition process in the discrete MIMO system has the following simple form:

$$\boldsymbol{F} = -(\boldsymbol{B}^{+} + (\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{+}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}}\boldsymbol{B}^{\perp})\boldsymbol{A}. \qquad (47)$$

For $n = 3r$ ($L = 2$), the formula becomes more complicated:

$$\boldsymbol{F} = -(\boldsymbol{B}^{+} + ((\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{+} + ((\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{+} \times$$
$$\times((\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp}(\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}} \times$$
$$\times(\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp\mathrm{T}}((\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}}\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B})^{\perp\mathrm{T}}) \times$$
$$\times\boldsymbol{B}^{\perp}\boldsymbol{A}\boldsymbol{B}^{\perp\mathrm{T}}\boldsymbol{B}^{\perp})\boldsymbol{A}. \qquad (48)$$

The distribution of the eigenvalues of the MIMO discrete system matrix (as per the circular Girko law [20]) with the dimension of the state space $n = 3600$ and $r = 900$ ($L = 3$) on the complex plane is provided in Fig. 2. The distribution of the eigenvalues for the closed-loop system is shown in Fig. 3. As can be seen, the accuracy of stabilization of the large randomized matrix's eigenvalues is $\sim 10^{-2}$.

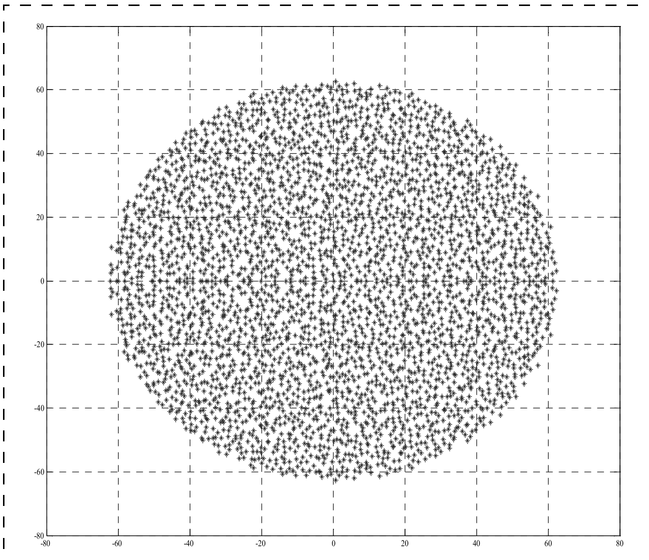

Fig. 2. The initial distribution of the MIMO system eigenvalues

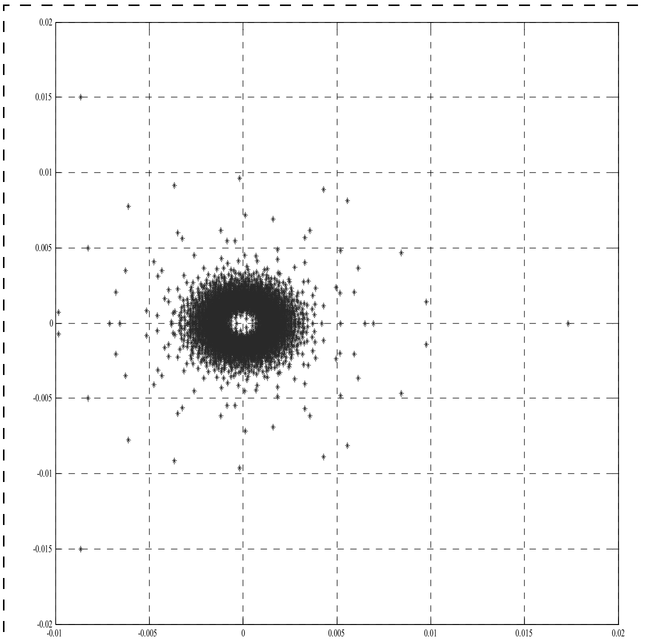

Fig. 3. Distribution of eigenvalues of the closed-loop MIMO system

5. Assessment of the computational burden

The computational burden of this method can be estimated on the basis of solving the precise poles placement problem at the points (−1, 0) and (−2, 0) on $\mathbb{C}$ for the system (1) with a pair of matrices [16]

$$\boldsymbol{A} = \begin{pmatrix} \mathrm{round}(\mathrm{randn}(r, r)) & 0_{r\times r} \\ \hline \multicolumn{2}{c}{\mathrm{randn}(r, 2r)} \end{pmatrix}, \ \boldsymbol{B} = \mathrm{randn}(2r, r).$$

Here "round" is a rounding operation; randn($2r \times r$) is the size of the submatrix $2r \times r$, and randn($r \times r$) is the size of the submatrix $r \times r$, whose elements are distributed in accordance with the pseudonormal law.

The results of the study for the MIMO system are presented by a diagram in Fig. 4. The abscissa axis represents the system's (1) n-dimensional state space, and the ordinate axis represents the time T_0 of the problem to be solved in the Matlab environment using a computer equipped with an Intel® Core™2 Quad CPU of 2.66 HGz with 3.25 Gbyte of RAM. For $n \leqslant 100$, on the basis of statistical tests of 1000 samples, the following approximating polynomial was obtained:

$$T_0(n) \approx 7{,}5 \cdot 10^{-7}n^2 - 5{,}6 \cdot 10^{-6}n + 2{,}6 \cdot 10^{-4}.$$

In all tests, the pole placement error is within the range of $10^{-14}...10^{-9}$; that is, it is a negligible quantity.

It should be noted that, with the help of Matlab procedure *place*, in which the Kautsky-Nichols-Van Dooren method [21] is implemented, a similar problem can only be solved up to $n = 500$. At the same time, the computational burden, when $n > 100$, will be higher than previously attained values when $n > 500$.

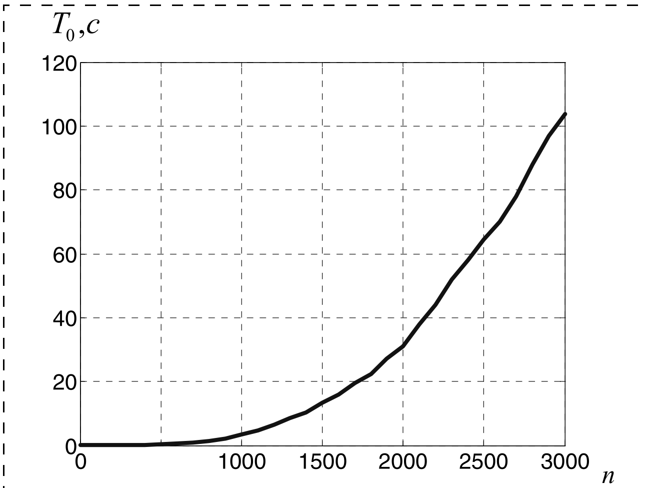

Fig. 4. Computational burden of the method

6. Conclusion

In this paper, an efficient method of full placement of the poles in linear MIMO systems has been proposed. The method is based on decomposition of the original model of the initial system determined in the state space. The method does not require the solution of a special matrix equation, it has the same form for continuous and discrete cases of the system model representation, it has no limitations with respect to the algebraic and geometric multiplicity of the specified poles, and it can be easily implemented in a Matlab software environment. Examples of solutions of precise pole placement problems for various MIMO systems, including systems with a state space of up to a few thousand, are presented. A comparative assessment of the computational burden has been carried out, demonstrating the advantage of the method in relation to the widely known Kautsky-Nichols-Van Dooren method.

Taking into account the duality of control and observation problems for linear MIMO systems, by using the proposed method it is easy to obtain formulas for the modal synthesis of the state observer.

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