DOI: 10.17587/mau.23.637-642

D. S. Fominykh, dm_fominyh@mail.ru, **A. F. Rezchikov, V. A. Kushnikov, V. A. Ivaschenko, A. S. Bogomolov,**
Saratov Science Center of RAS. Institute of Precision Mechanics and Control of RAS,
Saratov, 410028, Russian Federation,
V. A. Trapeznikov Institute of Control Sciences of RAS, Moscow, 117997, Russian Federation

Corresponding author: **Fominykh D. S.,** Cand. of Sc., Senior Researcher, Saratov Science Center of RAS. Institute of Precision Mechanics and Control of the Russian Academy of Sciences, Saratov, 410028, Russian Federation,
e-mail: dm_fominyh@mail.ru

Accepted on August 31, 2022

The Models and Algorithms for Product Quality Control in Welding by Robotic Technological Complexes

Abstract

The article discusses models and algorithms that allow quality control in welding by robotic technological complexes. The difference of this approach to solving the problem lies in the use of the calculus of variations and the definition of action plans that ensure the minimum deviation of the actual values of the actual values of the process quality indicators from the given values. The input data of the model are the target values of the quality indicators of the technological process, their actual values for a certain period and lists of measures to improve the quality of the process. As a measure of the deviation of quality indicators, objective functions were considered, which are minimized when solving the problem. An approach is considered on the example of arc welding of metal structures using Kawasaki robotic technological complexes with C40 controllers. The area of application of the developed software is the control systems of robotic complexes.

Keywords: *algorithm, mathematical model, quality control, robot, welding*

Acknowlegements: The research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (themes № FFNM-2022-0010 and № FMRN-2021-0001)

For citation:

Fominykh D. S., Rezchikov A. F., Kushnikov V. A., Ivaschenko V. A., Bogomolov A. S. The Models and Algorithms for Product Quality Control in Welding by Robotic Technological Complexes, *Mekhatronika, Avtomatizatsiya, Upravlenie*, 2022, vol. 23, no. 12, pp. 637—642.

DOI: 10.17587/mau.23.637-642

УДК 007:159.955

DOI: 10.17587/mau.23.637-642

Д. С. Фоминых, канд. техн. наук, ст. науч. сотр., dm_fominyh@mail.ru,
А. Ф. Резчиков, д-р техн. наук, проф., член-корр. РАН, **В. А. Кушников,** д-р техн. наук, проф.,
В. А. Иващенко, д-р техн. наук, проф., **А. С. Богомолов,** д-р техн. наук, доц.,
Институт проблем точной механики и управления — обособленное структурное подразделение
Федерального государственного бюджетного учреждения науки
Федерального исследовательского центра "Саратовский научный центр РАН", Саратов,
Институт проблем управления им. В. А. Трапезникова РАН, Москва

Модели и алгоритмы управления качеством продукции при сварке роботизированными технологическими комплексами*

Рассматриваются модели и алгоритмы, позволяющие осуществить управление качеством сварки роботизированными технологическими комплексами. Отличие описываемого подхода к решению задачи заключается в использовании математического аппарата вариационного исчисления и определении планов действий, обеспечивающих минимальное отклонение фактических значений показателей качества процесса от заданных значений. Входными данными модели являются целевые значения показателей качества технологического процесса, их фактические значения за определенный период и перечни мероприятий по повышению качества процесса. Для вычисления отклонения показателей качества рассмотрены целевые функции, которые минимизируются при решении задачи. Рассмотрен подход на примере дуговой сварки металлоконструкций с использованием робототехнических комплексов Kawasaki с контроллерами C40. Область применения разработанного программного обеспечения — системы управления робототехническими комплексами.

Ключевые слова*: алгоритм, математическая модель, управление качеством, робот, сварка*

*Работа выполнена в рамках государственного задания Министерства науки и высшего образования Российской Федерации (тема № FFNM-2022-0010 и тема № FMRN-2021-0001)

Introduction

Due to the rapid development of technology and growing competition, industrial enterprises' main problems are to ensure the quality of products. The technological process of arc welding of metal structures using robotic technological complexes (RTC) is not an exception. The primary purpose of using welding robots is to increase labour productivity and product quality. An insufficient level of control during the process for any reason increases the risk of defective products.

The RTC is an integrated system with many different components. A typical RTC of arc welding consists of manipulators equipped with welding equipment: power source, wire feed unit, cooling unit, welding torch, etc (Fig. 1). The complex also is equipped with safety devices (fencing, emergency stop buttons, laser barriers). Control of the RTC is carried out by the operator through a portable console connected to the controller. RTC control requires taking into account many parameters, both quantitative and qualitative types.

To date, various control systems are being introduced into the technological process of welding with the help of RTC. For example, in [1,2] the main attention is paid to solving the problem of tracking the welding path by methods of automatic identification using computer vision, controlling the wire feed speed depending on the arc current in real time.

In paper [3], using laser technologies, contactless collection of data on the parameters of the weld is carried out. This is achieved through processing sensory feedback data and allows real-time correction of robot paths to compensate for deviations in the position of parts.

Many control systems focus attention on the quality of friction stir welding [4—6]. Here, on the basis of iterative algorithms, the welding trajectory is planned, and the efficiency of the joint is estimated depending on such parameters as the gap width, the angle of movement, and the penetration depth.

Known methods [7] which use a camera on the manipulator to correct the programmed trajectory of the robot in an autonomous mode.

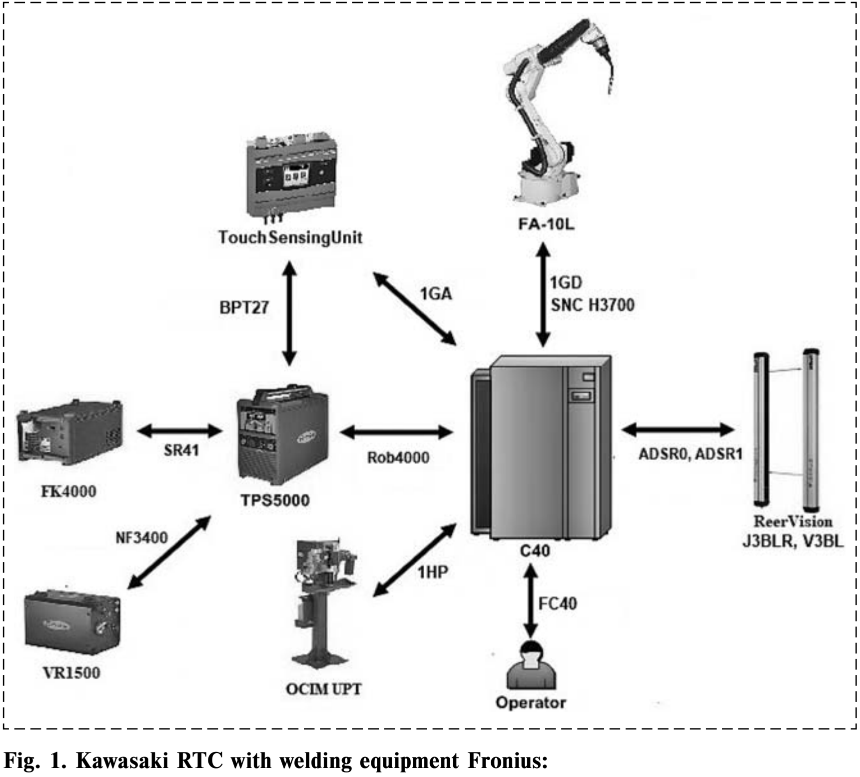

Fig. 1. Kawasaki RTC with welding equipment Fronius:
TPS5000 — the power source Fronius TransPulseSynergic5000, C40 — controller Kawasaki C40 series, FA-10L — robot manipulator Kawasaki FA-10L, 1GA — the central control unit of the controller; 1HP is the control unit of the servodrivers; FC40 — the multifunctional operator console; 1GB is the engine control unit of the axes of the manipulator; Rob4000 is the interface for communication with the welding equipment; VR1500 — wire feed unit; OCIM UPT — torch cleaning station; ReerVision — safety bareer; SR41, NF3400, BPT27, SNC H3700 — connection units; FK4000 — torch cooling unit

In [8], the problem of matching several motions of welding devices was proposed. Here, the authors obtained an analytical solution for finding the Cartesian positions of the caterpillar and the working body of the robot, which were used to create the articulation angles of the arm using inverse kinematics by holding the six-axis lever in a position of good maneuverability.

In addition, a number of researchers propose a remotely controlled welding scheme, which makes it possible to transform the knowledge of a human welder into a welding robot [9]. This is achieved by equipping the industrial robot arm with sensors to monitor the welding process, including a compact 3D weld pool surface measurement system and an additional camera that provides a direct view of the workpiece. The article [10] presents a genetic algorithm for the traveling salesman problem used to determine the sequence of welding tasks. A random key genetic algorithm used to solve a sequence of multi-robot welding tasks: multi-robot welding. In work [11], the problem of preventing critical combinations of events leading to an accident is solved, each of which separately does not lead to an accident.

Simultaneously, it is necessary to pay more attention to the optimization welding process's operational control, taking into account all the technological process parameters of the and the human factor and disturbances that affect the products' quality. These circumstances determine the relevance and practical significance of this article, which contains the development of models and algorithms for controlling the welding process in the RTC according to the criterion that minimizes production quality deviations.

Formulation of the problem

To develop mathematical models and algorithms that allow on the time interval $[t_0, t_1]$ for any permissible values of the vector of states $\mathbf{v}(t) \in \mathbf{V}$ of the environment to find the vector of control actions on the RTC $\mathbf{u}(t) \in \mathbf{U}$ to minimize the objective function:

$$Q(t) = \int_{t_0}^{t_1} \sum_{i=1}^{n} (X_i^*(t) - X_i(t))^2 \omega_i dt \to \min, \quad (1)$$

under limitations:

$$\begin{cases} L_j(\mathbf{t}, \mathbf{v}, \mathbf{v}', \mathbf{u}, \mathbf{u}') \geqslant 0, & j = 1, \ldots, m_1, \\ L_j(\mathbf{t}, \mathbf{v}, \mathbf{v}', \mathbf{u}, \mathbf{u}') < 0, & j = m_1 + 1, \ldots, m_2 \end{cases}$$

and border conditions:

$$\begin{cases} L_j^{(t_0)}(t, \mathbf{v}, \mathbf{v}', \mathbf{u}, \mathbf{u}') = 0, & j = m_3, \ldots, m_4, \\ L_j^{(t_1)}(t, \mathbf{v}, \mathbf{v}', \mathbf{v}, \mathbf{v}') = 0, & j = m_4 + 1, \ldots, m_5 \end{cases}$$

where $\tilde{X}_i$, X_i, $i = 1, 2, \ldots, n$ are the target and actual indicators of the welding process quality in the RTC, respectively; ω_i is the i-th indicator's weighting coefficient $m_1 \ldots n_5$ are known constants.

Taking into account the assumptions made, the physical meaning of the problem consists in choosing from existing activities plans such a plan, in the implementation of which the weighted sum of deviations of the main quality indicators from their target values will be the smallest which will lead to a decrease in damage from low-quality products.

We formulated the main indicators of the quality of the welding process using RTC in [12, 13], for example, the number of defective beams per 100 items, the average length of bad welds per unit of production, the average deviation of welding arc voltage, etc. There, the authors propose a solution to the problem (1) using a system dynamics model that allows constructing the differential equations for the primary phase variables, taking into account the positive and negative rate of the velocity of the variables, which includes all the factors causing the growth of the variables.

The application of this approach made it possible to evaluate quality indicators at different time intervals and to control the RTC welding process in accordance with the quality criterion.

Despite several apparent advantages, dynamic modelling has some limitations associated primarily with the accuracy of modelling and the complexity of estimating its error. This article proposes to develop a mathematical model to obtain an analytical solution to the problem (1). Because the answer to the problem is to find the minimum of the objective function, it is advisable to use the classical apparatus of the calculus of variations based on finding the extremum of functionals.

We can consider problem (1) as a variational problem of finding a conditional extremum, and to solve it, it is necessary to find the extremal of the functional:

$$J = \int_{t_0}^{t_1} F(X_1(t), X_2(t), \ldots, X_n(t)) dt,$$

$$F(X_1(t), X_2(t), \ldots, X_n(t)) = \sum_{i=1}^{n} (\tilde{X}_i - X_i(t))^2 \omega_i.$$

Development of the mathematical model

To solve the problem, we will use the Lagrange multipliers method. According to [14], it is necessary to introduce new functional:

$$J_1 = \int_{t_0}^{t_1} \tilde{F}(X_1(t), X_2(t), \ldots, X_n(t)) dt,$$

$$\tilde{F}(X_1(t), X_2(t), \ldots, X_n(t)) =$$
$$= F(X_1(t), X_2(t), \ldots, X_n(t)) + \sum_{j=1}^{m} \lambda_j \varphi_j,$$

where λ_j, $j = 1, 2, \ldots, m$ are Lagrange multipliers, φ_j are constraint equations.

The following system of equations specifies the necessary conditions for the presence of an extremum:

$$\begin{cases} \dfrac{\partial \tilde{F}}{\partial X_i} = 0, & i = 1, 2, \ldots n; \\ \varphi_j = 0, & j = 1, 2, \ldots m. \end{cases} \quad (2)$$

As the equations of connection, we use the approximated functional relationships between the indicators obtained in [13]. Then the system of equations (2) will take the following form:

$$|\mathbf{A}| * |\mathbf{\Phi}| = 0$$

$$\mathbf{A} = \begin{vmatrix} -2\tilde{X}_1\omega_1 + 2X_1\omega_1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \\ + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + 1{,}68\lambda_{33}X_1 - 0{,}59\lambda_{33} \\ -2\tilde{X}_2\omega_2 + 2X_2\omega_2 + 2{,}46X_2\lambda_5 - 3{,}22\lambda_5 + \\ + 0{,}15X_2^2\lambda_{13} - 1{,}52X_2\lambda_{13} + 4{,}74\lambda_{13} \\ \ldots \\ -2\tilde{X}_{17}\omega_{17} + 2X_{17}\omega_{17} - 0{,}93X_{17}^2\lambda_8 + \\ + 4{,}24X_{17}\lambda_8 - 0{,}18\lambda_8 + 0{,}87X_{17}^2\lambda_{19} - \\ - 2{,}12X_{17}\lambda_{19} + 1{,}2\lambda_{17} + \lambda_{28} \\ -2\tilde{X}_{18}\omega_{18} + 2X_{18}\omega_{18} + \lambda_{29} + \lambda_{30} + \\ + \lambda_{31} + \lambda_{32} + \lambda_{33} + \lambda_{34} \end{vmatrix};$$

$$\mathbf{\Phi} = \begin{vmatrix} X_1 - 0{,}23X_3^2 + 0{,}29X_3 - 0{,}19 \\ X_1 + 0{,}36X_{11}^3 - 1{,}74X_{11}^2 + 0{,}09X_{11} - 1{,}13 \\ \ldots \\ X_{18} + 0{,}84X_1^2 - 0{,}59X_1 + 1{,}08 \\ X_{18} + 1{,}08X_4^3 - 1{,}14X_4^2 + 0{,}74X_4 - 0{,}98 \end{vmatrix}.$$

Matrixes **A** and **Φ** are presented in compressed forms to avoid bulkiness.

Having solved the equation system (2) we obtain the extremum points of the functional J_1. In order to determine whether the conditional minimum or conditional maximum has a functional J_1 at these points, methods [14] were used.

To achieve the obtained values of quality indicators, it is necessary to implement control actions. Based on experience, it is known that the operational dispatch personnel uses a limited number of standard activities plans $\{u_1, u_2, \ldots u_m\}$ that are formed by experts.

Each of these plans is presented as a frame:

<*name*; (Act_1; R_ex_1; Pl_1; T_1); ... (Act_M; R_ex_M; Pl_M; T_M)>.

Slots:

name is the name of the plan;

Act_i is a description of *i*-th activity of program;

R_ex_i contains information on who is responsible for the implementation of the *i*-th activity of the plan;

*Pl*м is the location of the *i*-th activity;

T_i is time for completing (frequency) of the *i*-th event, $i = 1, 2, \ldots M$.

Based on expert estimates, we approximated the dependences of quality indicators $X_1, X_2, \ldots, X_{18}$ on the implementation of each activities plan in the form of polynomials of the second degree:

$$X_i(u,t) = \begin{cases} a_1^{(i)}t^2 + b_1^{(i)}t + c_1^{(i)}, \text{ если } u = u_1; \\ a_2^{(i)}t^2 + b_2^{(i)}t + c_2^{(i)}, \text{ если } u = u_2; \\ \ldots \\ a_m^{(i)}t^2 + b_m^{(i)}t + c_m^{(i)}, \text{ если } u = u_m; \end{cases}$$

Thus, the desired vector of control action will be the activities plan that will provide the minor deviation $X_i(u,t)$ from $X_i^*(t)$. For this, we introduce the metric:

$$\Omega(u_k) = \sum_{i=1}^{18} \left| X_i(u_k,t) - X_i^*(t) \right| \omega_i .$$

The activities plan u^* will be a solution to the problem (1) if the following condition is satisfied for it:

$$\forall i \in [1,m] \Rightarrow \Omega(u_i) \geqslant \Omega(u^*).$$

Calculating $\Omega(u_k)$ sequentially for $k = 1, 2, \ldots, m$, we choose the plan that corresponds to the minimum value $\Omega(u_k)$. In the process of adapting the developed mathematical model the conditions for the functioning of a specific control system when choosing each of the above functions the area of its admissible values is analyzed and restrictions are imposed on the area of definition of the function. Before the stage of practical use of the model the calculation results are compared with the values of the simulated variables known from practice and if there are significant discrepancies, the model is corrected.

The model example

Let's illustrate the features of the application of the developed mathematical model by an example the technological process of welding using the Kawasaki RTC with C40 controllers and associated Fronius welding equipment.

To calculate the objective function, it is necessary to set the values of the weight coefficients. These coefficients are selected based on the experience of the operating dispatching personnel and determine the significance of each quality indicator. For RTC Kawasaki the specified coefficients are given in Table 1.

Further, by solving the system of equations (2), we obtain the importance of the indicators at which the function $Q(t)$ takes a minimal value, taking into account the given restrictions:

Table 1

The quality indicators for welding in robotic technological complexes

Quality indicator	$\tilde{X}_i$	ω_i
X_1	0,07	0,14
X_2	0,5	0,07
X_3	0,1	0,02
X_4	0,5	0,05
X_5	0,98	0,03
X_6	0,55	0,04
X_7	0,85	0,02
X_8	0,9	0,06
X_9	0,3	0,02
X_{10}	0,25	0,07
X_{11}	0,2	0,03
X_{12}	0,3	0,07
X_{13}	0,6	0,04
X_{14}	0,97	0,06
X_{15}	0,3	0,06
X_{16}	0,6	0,04
X_{17}	0,95	0,08
X_{18}	0,9	0,1

$$\begin{gathered} X^* = \{X_1 = 0,075;\ X_2 = 0,613;\ X_3 = 0,29; \\ X_4 = 0,379;\ X_5 = 0,869;\ X_6 = 0,661; \\ X_7 = 0,773; X_8 = 0,849;\ X_9 = 0,275; \\ X_{10} = 0,318;\ X_{11} = 0,37;\ X_{12} = 0,303; \\ X_{13} = 0,688; X_{14} = 0,933;\ X_{15} = 0,215; \\ X_{16} = 0,512;\ X_{17} = 0,948;\ X_{18} = 0,955\} \end{gathered} \quad (3)$$

We need to find an activities plan that brings quality indicators to the values (3). Based on the control object's long-term observations, we formed the dependences of the indicators X_1, X_2, ..., X_{18} on the activities plans u_1, u_2, ..., u_6. For example, the reliance of the indicator X_1 ("The number of rejected beams per 100 units of production") from the activities plan looks as follows:

$$X_1(u,t) = \begin{cases} -0,003t^2 + 0,015t + 0,077, \text{ if } u = u_1; \\ -0,003t^2 + 0,01t + 0,095, \text{ if } u = u_2; \\ -0,017t + 0,137, \text{ if } u = u_3; \\ -0,003t^2 - 0,006t + 0,131, \text{ if } u = u_4; \\ -0,003t^2 - 0,0058t + 0,149, \text{ if } u = u_5; \\ -0,003t^2 + 0,0045t + 0,113, \text{ if } u = u_6 \end{cases}$$

We will find the minimum values of $\Omega(u_i)$ at different times to determine the best activities plan. We performed calculations on the interval $t = [0; 1]$, corresponding to a period of 1 year. We summarized the calculation results in Table 2. As you can see, on the time interval [0;0,3], the plan of activities u_5 is optimal; on the interval [0,4; 1], the optimal plan is u_6. Taking into account the proximity of the values $\Omega(u_5)$ and $\Omega(u_6)$ on the interval [0;0,3], we can take plan u_6 as the vector of control actions for the entire coming year.

Table 2

Selection the activities plan on time interval

t	$\Omega(u_1)$	$\Omega(u_2)$	$\Omega(u_3)$	$\Omega(u_4)$	$\Omega(u_5)$	$\Omega(u_6)$
0,1	0,023	0,034	0,102	0,017	0,073	0,026
0,2	0,026	0,093	0,151	0,027	0,053	0,029
0,3	0,052	0,063	0,122	0,027	0,044	0,046
0,4	0,003	0,103	0,161	0,086	0,102	0,016
0,5	0,082	0,122	0,131	0,115	0,171	0,065
0,6	0,072	0,005	0,112	0,056	0,102	0,035
0,7	0,101	0,024	0,171	0,043	0,161	0,006
0,8	0,111	0,014	0,171	0,007	0,142	0,065
0,9	0,072	0,093	0,082	0,115	0,102	0,114
1	0,121	0,054	0,082	0,086	0,122	0,095

You can see a fragment of the frame corresponding to this plan below:

<u_6; (*Make intermediate quality control of the welded seam*; *RTC operator*; *Assembly and welding workshop*; *Every shift*); (*Check the relevance of technological documentation at workplaces*; *Technologist*; *Assembly and welding shop*; *Daily*); (*Monitor the values of the welding current by the indicators of the power source during welding of the product*; *RTC operator*; *Assembly and welding shop*; *Every hour*); (*Carry out unscheduled maintenance of the RTC*; *Adjuster of welding equipment*; *Assembly and welding shop*; *Within a week*); (*Hire one more programmer*; *HR Inspector*; *Human Resources*; *Within a month*)>.

Thus, the implementation of the activities plan u_6 provides a minimum of the target function and, therefore, is a solution to problem (1).

The proximity of quality indicators after implementing the activities plan u_6 to the values of X_i* we can evaluate by the radar diagram in Fig. 2.

As you can see, the execution plan u_6 allows to increase the number of beams handed over to the quality control department from the first exit

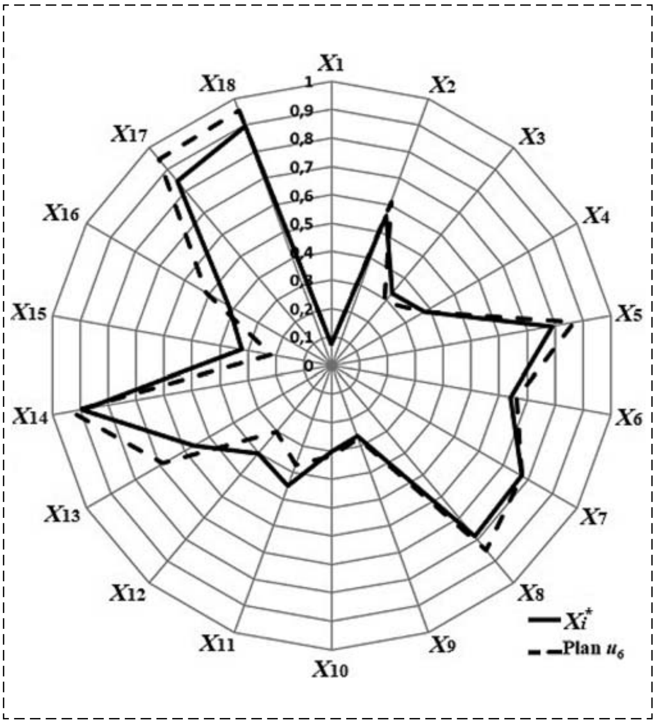

Fig. 2. Comparison of the calculated quality indicators and indicators after the implementation of the activities plan u$_6$

(X_{18} indicator) and to reduce the number of emergency stops of the RTC (X_3 indicator).

Conclusion

With the help of the mathematical support proposed in the article, it is possible to develop activities plan to ensure the minimum deviation of quality indicators from the target ones.

A feature of the model is the use of the classical apparatus of the calculus of variations, making it possible to increase modelling accuracy. The authors developed these models and algorithms for RTC Kawasaki control systems with C40 controllers; however, it is possible to replicate the model for machine-building enterprises using arc welding of various models RTC.

References

1. **Dinham M., Fang G** Autonomous weld seam identification and localization using eye-in-hand stereo vision for robotic arc welding, *Robotics and Computer-Integrated Manufacturing,* 2013, vol. 29, no. 5, pp. 288—301.

2. **Shen H. Y., Wu J., Lin T., Chen S. B.** Arc welding robot system with seam tracking and weld pool control based on passive vision, *International Journal of Advanced Manufacturing Technology,* 2008, vol. 39, no. 7—8, pp. 669—678.

3. **Agapakis J. E., Katz J. M., Friedman J. M., Epstein G. N.** Vision-aided robotic welding. An approach and a flexible implementation, *International Journal of Robotics Research*, 1990, vol. 9, no. 5, pp. 17—34.

4. **Guillo M., Dubourg L.** Impact & improvement of tool deviation in friction stir welding: weld quality & real-time compensation on an industrial robot, *Robotics and Computer-Integrated Manufacturing,* 2016, no. 39, pp. 22—31.

5. **Shultz E. F., Cole E. G., Smith C. B., Zinn M. R., Ferrier N. J., Pfefferkorn F. E.** Effect of compliance and travel angle on friction stir welding with gaps. Journal of Manufacturing Science and Engineering, *Transactions of the ASME*, 2010, vol. 132, no. 4, pp. 0410101—0410109.

6. **Wu J., Zhang R., Yang G.** Design and experiment verification of a new heavy friction-stir-weld robot for large-scale complex surface structures, *Industrial Robot*, 2015, vol. 42, no. 4, pp. 332—338.

7. **Ryberg A., Ericsson M., Christiansson A. K., Eriksson K., Nilsson J., Larsson M.** Stereo vision for path correction in off-line programmed robot welding, *In Proceedings of the IEEE International Conference on Industrial Technology*, 2010, pp. 1700—1705.

8. **Ahmad S., Luo S.** Coordinated motion control of multiple robotic devices for welding and redundancy coordination through constrained optimization in cartesian space, *IEEE Transactions on Robotics and Automation*, 1989, vol. 5, no. 4, pp. 409—417.

9. **Liu Y. K., Zhang Y. M.** Toward welding robot with human knowledge: a remotely-controlled approach, *IEEE Transactions on Automation Science and Engineering,* 2015, vol. 12, no. 2, pp. 769—774.

10. **Kim K. Y., Kim D. W., Nnaji B. O.** Robot arc welding task sequencing using genetic algorithms, *IIE Transactions* (Institute of Industrial Engineers), 2002, vol. 34, no. 10, pp. 865—880.

11. **Rezchikov A. F., Kushnikov V. A., Ivashchenko V. A., Fominykh D. S., Bogomolov A. S., Filimonyuk L. Y.** Prevention of critical events combination in robotic welding, *Journal of Machinery Manufacture and Reliability*, 2017, vol. 46, no. 4, pp. 370—379.

12. **Fominykh D. S., Kushnikov V. A., Rezchikov A. F.** Prevention unstable conditions in the welding process via robotic technological complexes, *MATEC Web of Conferences,* 2018, vol. 224, pp. 01045.

13. **Fominykh D. S., Kushnikov V. A., Rezchikov A. F.** Control of the welding process in robotic technological complexes using the system dynamics model, *Proceedings of the International Multi-Conference on Industrial Engineering and Modern Technologies*, FarEastCon, Vladivostok, 2019, pp. 8933981.

14. **Kalman D.** (2009). Leveling with Lagrange: an alternate view of constrained optimization*, Mathematics Magazine*, vol. 82, no. 3, pp. 186—196.