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Analysis of a Cart-Inverted Pendulum System with Harmonic Disturbances Based on its Criterion Matrix

Abstract

The control of an inverted pendulum is a classical benchmark control problem. Its dynamics resemble that of many real-world systems of interest like pendulous, missile launchers, segways, and many more. The control of this system is challenging as it is a highly unstable, highly non-linear, non-minimum phase system, and underactuated. Furthermore, the physical constraints on the track position also pose complexity in its control design. A great deal of nonlinearity is present inherently and as well as affected by the surrounding external disturbances. The paper presents an approach for analysis of a cart-inverted pendulum system with harmonic disturbances. The approach is based on the index of the criterion matrix of the system named a degeneration factor. The degeneration factor is constructed with the singular values of the criterion matrix of the system and allows us to find frequency range, where the system operates as a whole. A linear-quadratic regulator is used to control the cart-inverted pendulum system. The results are supported with an example.

Keywords: *cart-inverted pendulum, criterion matrix, Sylvester equation, degeneration factor, harmonic disturbance, linear-quadratic regulator*

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Анализ системы "маятник—тележка" при внешнем гармоническом воздействии на основе критериальной матрицы системы

Управление перевернутым маятником на тележке является классической задачей теории управления. Динамика перевернутого маятника схожа с динамикой многих реальных систем, представляющих практический интерес, таких как маятниковые системы, ракетные пусковые установки, сигвеи и многие другие. Управление такой системой является сложной задачей в силу неустойчивости и нелинейности системы. Кроме того, физические ограничения, накладываемые на систему, также усложняют процесс проектирования системы управления. Большая часть нелинейностей обусловлена как самой конструкцией системы, так и влиянием внешних возмущений различного характера. В статье представлен подход к анализу системы "маятник—тележка", функционирующей в условиях внешних гармонических воздействий. Предлагаемый подход основан на исследовании характеристического показателя критериальной матрицы системы, именуемого функционал вырождения. Функционал вырождения, сконструированный на спектре сингулярных чисел критериальной матрицы системы, используется как качественный показатель, позволяющий определить диапазон частот гармонического воздействия, на котором система функционирует как единое целое. Для управления системой перевернутого маятника на тележке используется линейно-квадратичный регулятор. Предлагаемый подход иллюстрируется примером.

Ключевые слова: *перевернутый маятник на тележке, критериальная матрица, уравнение Сильвестра, функционал вырождения, гармоническое возмущение, линейно-квадратичный регулятор*

Introduction

An inverted pendulum on a cart is an unstable nonlinear system that is often used to test the performance and effectiveness of control algorithms [1—9]. Many algorithms have been successfully applied to this model such as PID control [1, 2, 8], fuzzy logic control [3, 4], neural network [5], sliding mode control [6, 7], and linear—quadratic regulator (LQR) control [8, 9]. In these papers, the authors only presented the inverted pendulum is affected by the random noise or moved from the initial position to the equilibrium position without mentioning the effects of harmonic disturbances. Harmonic disturbances are common signals in practice [10, 11], they have great influences on the working process of MIMO systems [12]. During the working process of the MIMO system, there are many harmonic disturbances that cause instability and degeneration [12, 13]. For example, when the system moves on an undulating or vibrating surface.

In this paper, the authors present an approach for analysis of a cart-inverted pendulum system with harmonic disturbances. The approach is based on the index of the criterion matrix of the system named a degeneration factor [14—16]. The degeneration factor is constructed with the singular values of the criterion matrix of the system and allows us to find frequency range, where the system operates as a whole. A linear-quadratic regulator is used to control the cart-inverted pendulum system. The proposed approach consists of two steps. The first step is the criterion matrix construction. The second step is degeneration factor calculation with the further analysis.

The paper is laid out as follows. The problem formulation with the description of the researched model is given in Section 1. Then, the control design procedure for the inverted pendulum on a cart system is described and the methodology for the criterion matrix constructing is presented in Section 2. Thereafter, simulation results of the designed control system with harmonic disturbances are depicted. The cases of harmonic disturbances of different frequencies are considered. The researched system is analyzed with the degeneration factor of its criterion matrix. As the result, a frequency range, where the system operates as a whole, is defined. The paper is finished with some concluding remarks.

1. Problem formulation

The inverted pendulum is easily destabilized under the impact of disturbances on the position of cart and the angle of pendulum. Firstly, we consider the mathematical model that describes the inverted pendulum. The inverted pendulum model consists of a cart and a pendulum (Fig. 1). The position of the cart and the angle of the pendulum are able to be controlled. It supposes that frictions and moments of inertia are ignored. The nonlinear model of the Inverted Pendulum is constructed by using the Euler-Lagrange equation.

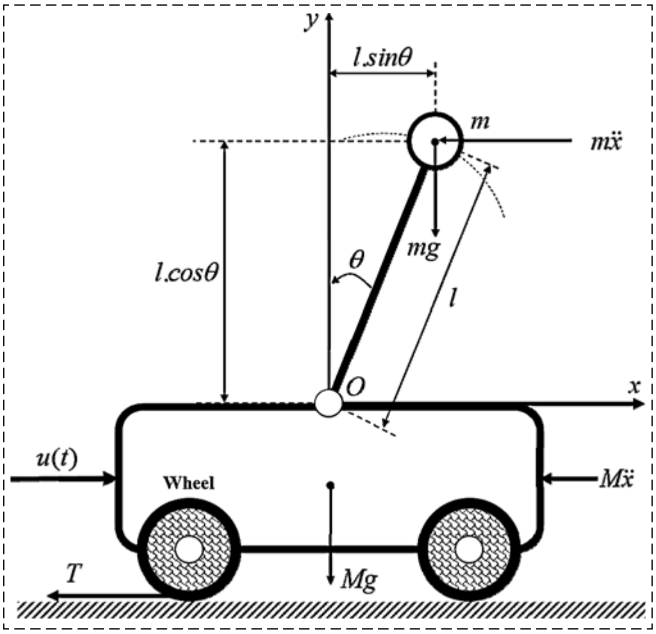

Fig. 1. Inverted pendulum on a cart

The kinetic energy of the pendulum can be calculated as

$$T_p = \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{2}ml^2\dot{\theta}^2. \qquad (1)$$

The kinetic energy of the cart satisfies the following expression:

$$T_c = \frac{1}{2}M\dot{x}^2. \qquad (2)$$

Then, the kinetic energy of the system is given by

$$T = T_c + T_p = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{2}ml^2\dot{\theta}^2. \qquad (3)$$

The potential energy of the system is described by the following expression

$$U = mgl\cos(\theta). \qquad (4)$$

Then, the energy of the inverted pendulum:

$$L = T - U = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos(\theta) + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos(\theta). \qquad (5)$$

Consider the Euler-Lagrange equation:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u - k\dot{x}; \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0. \end{aligned} \tag{6}$$

Combination of (5) and (6) results in:

$$\begin{cases} (M + m)\ddot{x} - ml\sin\theta\dot{\theta}^2 + ml\cos\theta\ddot{\theta} = u - k\dot{x}; \\ ml^2\ddot{\theta} - mgl\sin\theta = -ml\ddot{x}\cos\theta. \end{cases} \tag{7}$$

Then, the state-space model representation of the inverted pendulum can be described in the following form:

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = \dfrac{ml\sin x_3 x_4^2 - mg\cos x_3 \sin x_3 + u - kx_2}{M + m(\sin x_3)^2}; \\ \dot{x}_3 = x_4; \\ \dot{x}_4 = \dfrac{ml\sin x_3 \cos x_3 x_4^2 - -(M+m)g\sin x_3 + u\cos x_3 - kx_2\cos x_3}{-(M + m(\sin x_3)^2)}, \end{cases} \tag{8}$$

where $x_1 = x;\ x_2 = \dot{x};\ x_3 = \theta;\ x_4 = \dot{\theta}$.

Equations (8) get the balance point $x_1 = 0$; $x_2 = 0$; $x_3 = 0$; $x_4 = 0$. The linearized state equation of the Inverted Pendulum has the form as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix} u, \tag{9}$$

where x — distance (m); $\dot{x}$ — velocity (m/s); $\ddot{x}$ — acceleration (m/s^2); θ — angular (rad); $\dot{\theta}$ — angular velocity (rad/s); $\ddot{\theta}$ — angular acceleration (rad/s^2); F — force (N); k — friction coefficient (Nm/A); g — gravity acceleration (m/s^2); M — mass of the cart (kg); m — mass of pendulum (kg); l — length of the pendulum (m).

From the differential equations describing the inverted pendulum (8), if a disturbance signal acts on the position of the cart or the angle of the pendulum, it will lead to instability or degeneration of the system. Then, the aim of the paper is to analyze the system behavior for the case of harmonic disturbances of different frequencies and find a frequency range, where the system operates as a whole.

In the next section, we will design the control law and describe the methodology for the criterion matrix constructing.

2. Control design method

A. Control law design for inverted pendulum on a cart

The idea of control law design for MIMO systems with harmonic disturbances is proposed in the works [14—16]. The algorithm will be applied to the researched inverted pendulum system. Consider a state-space model representation of the system in the following form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t); \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{10}$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix}$;

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$; **u**, **x**, **y** are control input, state vector, and output, respectively; $\mathbf{x} = [x_1\ \ x_2\ \ x_3\ \ x_4]^T \in R^4$; $\mathbf{u}, \mathbf{y} \in R^2$.

We assume that the harmonic disturbances affected on the inverted pendulum is given as

$$\mathbf{g}(t) = \boldsymbol{\psi}\sin\boldsymbol{\omega}t, \tag{11}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ are the amplitudes and the frequencies of harmonics disturbance, respectively. The control diagram of the inverted pendulum on a cart system is shown in Fig. 2.

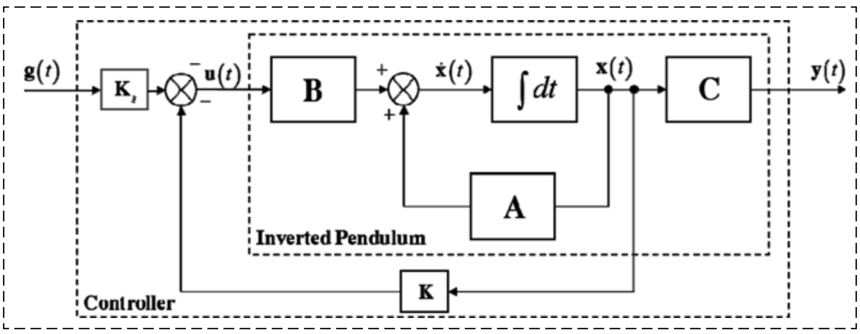

Fig. 2. The controller of the inverted pendulum on a cart

The control algorithm is proposed in the following form:

$$\mathbf{u}(t) = \mathbf{K_g g}(t) - \mathbf{Kx}(t), \quad (12)$$

where $\mathbf{K_g}$ is the feedforward scaling factor, and $\mathbf{K}$ is the gain of negative feedback.

The combination of dynamic model (10) and the control algorithm (12) allows us to get the closed-loop system:

$$\dot{\mathbf{x}}(t) = \mathbf{Fx}(t) + \mathbf{Gg}(t);\ \mathbf{x}(0), \mathbf{y}(0) = \mathbf{Cx}(t), \quad (13)$$

where $\mathbf{F} = \mathbf{A} - \mathbf{BK}$, $\mathbf{G} = \mathbf{BK_g}$. The resulting matrices of the system are used to obtain the criterion matrix of the inverted pendulum on a cart system.

B. Criterion matrix constructing

The state-space model representation for the inverted pendulum on a cart system with controller is given by:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{Fx}(t) + \mathbf{Gg}(t); \mathbf{x}(0); \\ \mathbf{y}(t) = \mathbf{Cx}(t), \end{cases} \quad (14)$$

where $\mathbf{x}$, $\mathbf{g}$, $\mathbf{y}$ are the state vector, the input vector, and the output vector, respectively; $\mathbf{x} \in R^4$, $\mathbf{g}$, $\mathbf{y} \in R^2$; $\mathbf{F}$, $\mathbf{G}$, $\mathbf{C}$ are the state matrix, the input matrix, and the output matrix respectively, where $\mathbf{F} \in R^{4\times4}$, $\mathbf{G}$, $\mathbf{C}^T \in R^{2\times4}$.

An exogenous disturbance $\mathbf{g}(t)$ is considered in the following form

$$\dot{\mathbf{z}}(t) = \mathbf{Ez}(t);\ \mathbf{z}(0);\ \mathbf{g}(t) = \mathbf{Pz}(t), \quad (15)$$

where $\mathbf{z} \in R^2$ is the state vector of the disturbance model; $\mathbf{E}$, $\mathbf{P}$ are the state matrix and the output matrix, the matrix $\mathbf{P}$ satisfies the condition $\mathbf{PP}^T = \mathbf{I}$, here $\mathbf{I}$ is the identity matrix ($\mathbf{I} \in R^{4\times4}$).

Assume, the exogenous disturbance (15) be a harmonic disturbance with the frequency ω. Then, we have representation of matrix $\mathbf{E}$ in the following form

$$\mathbf{E} = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}. \quad (16)$$

The criterion matrix of the system with harmonic disturbance [16] can be given by

$$\mathbf{N} = \mathbf{CT}(\omega), \quad (17)$$

where $\mathbf{T}$ is the similarity matrix, that satisfies the Sylvester matrix equation [14-15] as

$$\mathbf{TE} - \mathbf{FT} = \mathbf{GP}. \quad (18)$$

The singular value decomposition (SVD) [17-19] of the criterion matrix of the system is used to calculate its degeneration factor [14] in the following form

$$J_{D_1} = \frac{\alpha_{\min}}{\alpha_{\max}} = \frac{\alpha_1}{\alpha_2}, \quad (19)$$

where α_{min} and α_{max} are the minimum and the maximum singular values of the criterion matrix respectively. The degeneration factor reflects the behavior of the system and allow us to find frequency range, where the system operates as a whole.

Example

Consider the inverted pendulum on a cart system with the following parameters: $M = 2$(kg), $m = 1$(kg), $l = 0.5$(m), $g = 9.81$(m/s^2). In the simulation, we use Linear—quadratic regulator (LQR) controller. The block diagram of the system with external harmonic disturbance is shown in Fig. 3. We can change the frequencies of inputs on two channels: the position of the card and the angle of the pendulum.

The degeneration factor for the inverted pendulum on a cart system with LQR controller is illustrated in Fig. 4. Obviously, the system operates as a whole, when the frequency of harmonic disturbances more than $35s^{-1}$.

Fig. 5 shows the behavior of the inverted pendulum with LQR controller without harmonic disturbances. The inverted pendulum moves from the

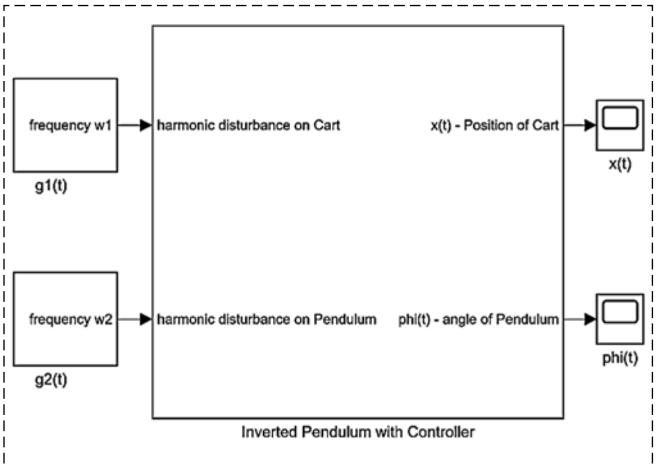

Fig. 3. The simulation model of the inverted pendulum on a cart

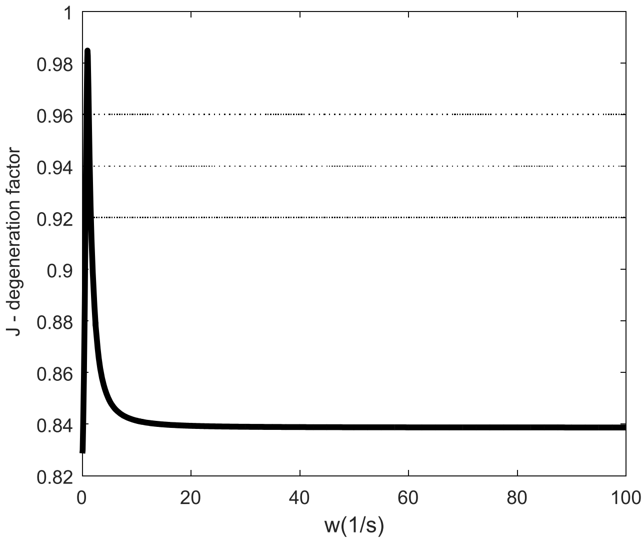

Fig. 4. Degeneration factor of the Inverted Pendulum

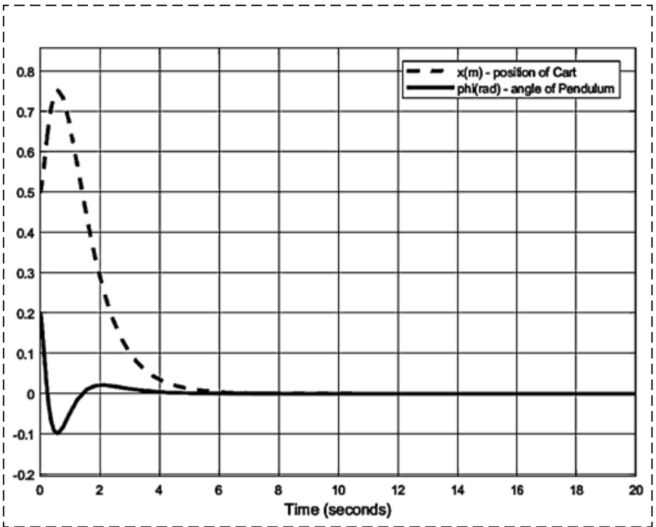

Fig. 5. The behavior of the system without harmonic disturbance

initial point (x = 0.5; θ = 0.2) to the equilibrium point (x = 0; θ = 0) with no setting error.

Fig. 6—8 show the inverted pendulum on a cart system with harmonic disturbances affected on the position of the cart and the angle of the pendulum. Fig. 6 illustrates the case frequencies of harmonic disturbances $\omega = 0.1s^{-1}$, the pendulum and the cart have big oscillations at the equilibrium point (θ = 0). When we change frequencies of harmonic inputs to $\omega = 10s^{-1}$, then the oscillations are decreased (Fig. 7). When frequencies of harmonic disturbances are $\omega = 100s^{-1}$ (Fig. 8) the inverted pendulum and the cart operates as a whole at equilibrium point (x = 0; θ = 0). The results correspond to the degeneration factor data.

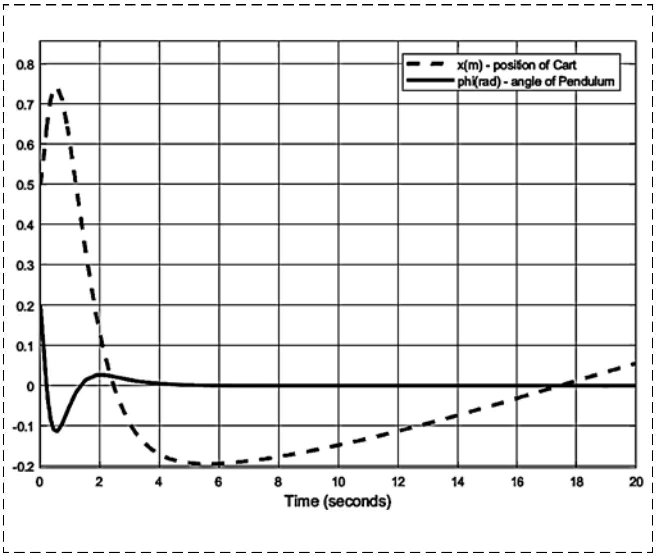

Fig. 6. The behavior of the system with the frequency of harmonic disturbance $\omega = 0.1s^{-1}$

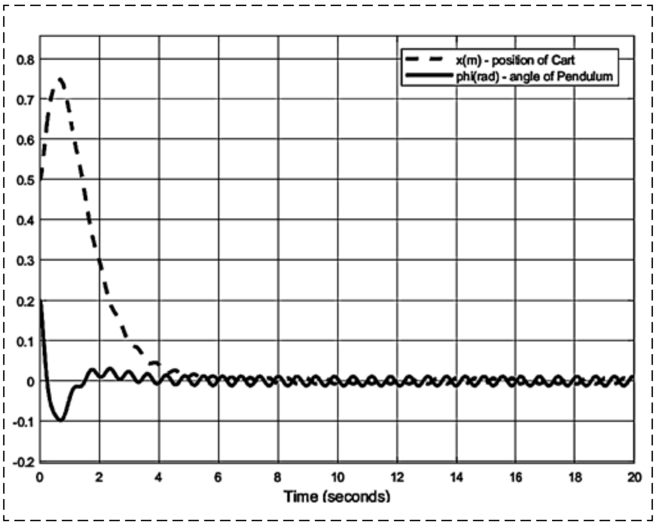

Fig. 7. The behavior of the system with the frequency of harmonic disturbance $\omega = 10s^{-1}$

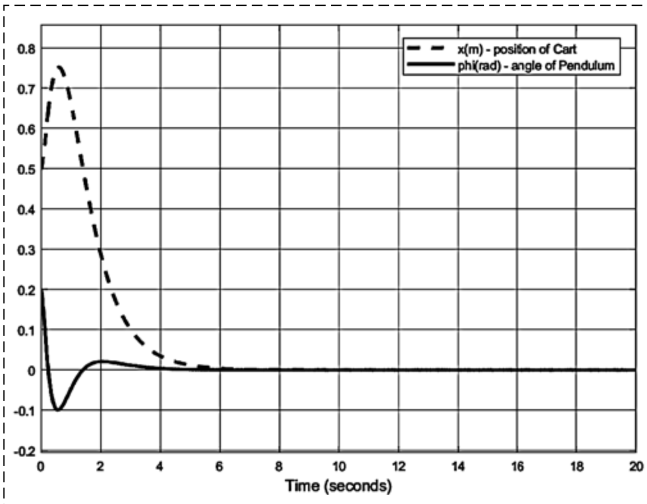

Fig. 8. The behavior of the system with the frequency of harmonic disturbance $\omega = 100s^{-1}$

Conclusion

The paper presents an approach for analysis of an inverted pendulum on a cart system with harmonic disturbances. It is proposed to use the degeneration factor of the criterion matrix of the system as a tool for analysis. The degeneration factor is constructed with the singular values of the criterion matrix of the system and allows us to find frequency range, where the system operates as a whole. Also, the approach can be applied to the linear multidimensional systems with harmonic disturbances.

In the future, the authors are going to expand the results to the case of double inverted pendulum system.

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